The language and series of Hammersley type processes

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Acks

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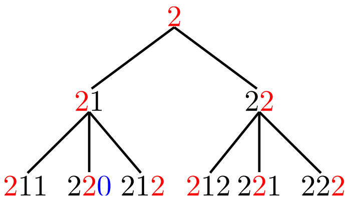
Summary for the technically-minded



- Study the grammatical complexity/formal power series of (generalization of) a model from the theory of interacting particle systems, the Hammersley process
- $k = 1$: $L^1_{HAM} = 1\{0,1\}^*$.
- $k \geq 2$: explicit form for L^k_H: DCFL, nonregular.
- Hammersley interval process: two languages, one equal to L^k_H, other explicit form, non-CFL (via Ogden).
- Algorithm for formal power series $\Rightarrow$ experiments, determining the value of a constant believed to be Φ.

"Story" for the Conceptually-minded

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- This paper: Attempt to prove this conjecture via formal power series. Made (baby) first-steps.

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- The (classical) Ulam-Hammersley problem.
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- The golden ratio conjecture and a "physics-like" argument.
- This paper: Attempt to prove this conjecture via formal power series. Made (baby) first-steps.
- This talk: One result, One proof, one algorithm, one experiment.

Starting Point: *Longest Increasing Subsequence*

3 2 5 7 1 6 9

Patience sorting.

Another (greedy, also first-year) algorithm:

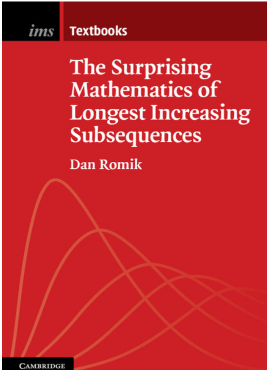
> Start (greedily) building decreasing piles. When not possible, start new pile.

Size of LIS = # of piles in patience sorting.

The Ulam-Hammersley problem (for random permutations)

What is the LIS of a random permutation ?

$$E_{\pi \in S_n}[LIS(\pi)] = 2\sqrt{n} \cdot (1 + o(1)).$$



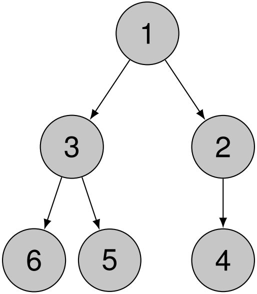
- Logan-Shepp (1977), Veršik-Kerov (1977), Aldous-Diaconis (1995)
- Very rich problem. Connections with nonequilibrium statistical physics and Young tableaux
- Also for intervals: Justicz, Scheinerman, Winkler (AMM 1990): random intervals on [0,1].

From (increasing) sequences to heaps

Byers, Heeringa, Mitzenmacher, Zervas (ANALCO'2011)

Sequence of integers A is heapable if it can be inserted into binary heap-ordered tree (not necessarily complete), always as leaf nodes.

Example: 1 3 2 6 5 4 Counterexample: 5 1 . . .



The Ulam-Hammersley problem for heapable sequences

- Simplest version trivial: $LHS(\pi) = n - o(n)$ (Byers et al.)
- (Dilworth, patience sorting): $LIS(\pi) =$ minimum number of decreasing sequences in a partition of π.

$HEAPS_k =$ minimum number of k-heapable sequences in a partition of π into such seqs.

Ulam-Hammersley problem for heapable sequences:

What is the scaling of $E_{\pi \in S_n}[HEAPS_k(\pi)]$, $k \geq 2$?

A beautiful conjecture

For $k \geq 2$ there exists $\lambda_k > 0$ such that

$$\lim_{n \to \infty} \frac{E[HEAPS_k(\pi)]}{ln(n)} = \lambda_k$$

Moreover

$$\lambda_2 = \frac{1 + \sqrt{5}}{2}$$

is the golden ratio.

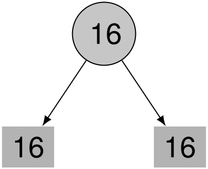
Status of the conjecture

- Some partial results.
- "Physics-like" nonrigorous argument, includes prediction for value of constant λ_k.
- Computations corroborated by experiments, "experimental mathematics" paper in progress.
- Follow-up work: Basdevant et al. (2016, 2017) rigorously establishes logarithmic scaling, but not the value of the constant.

Theorem: The "Patience heaping" algorithm correctly computes the value of parameter $Heaps_k(\pi)$.

Patience heaping

16, 25, 18, 2, 4, 35, 3, 7, 32, 9, 20



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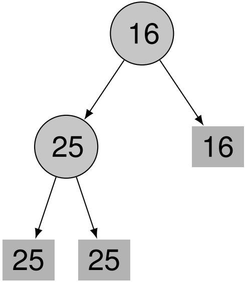

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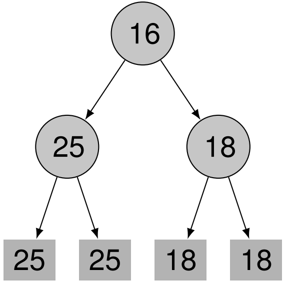

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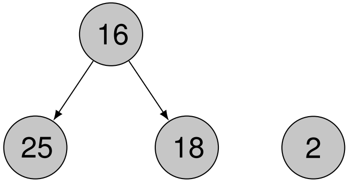

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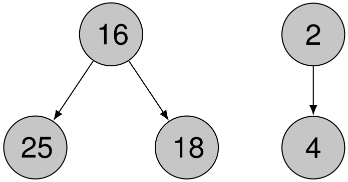

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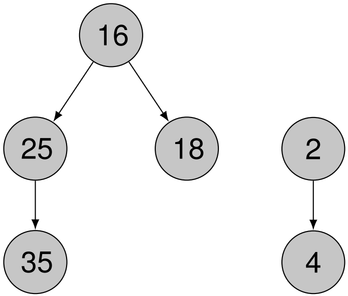

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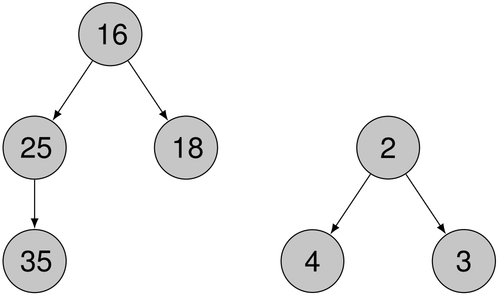

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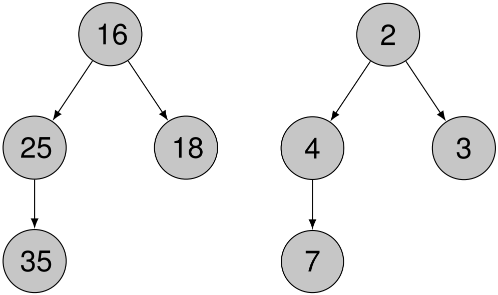

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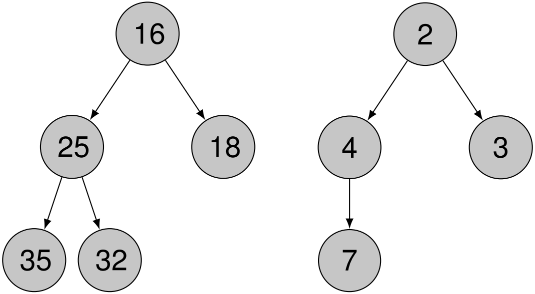

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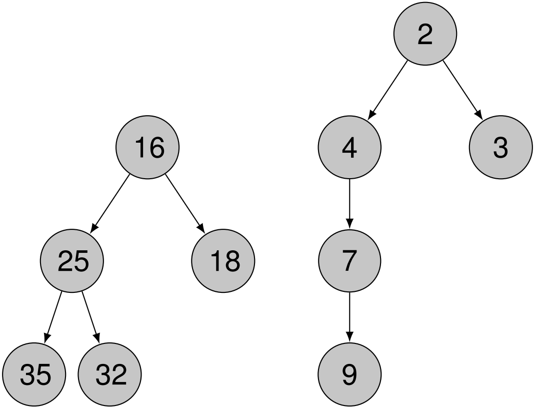

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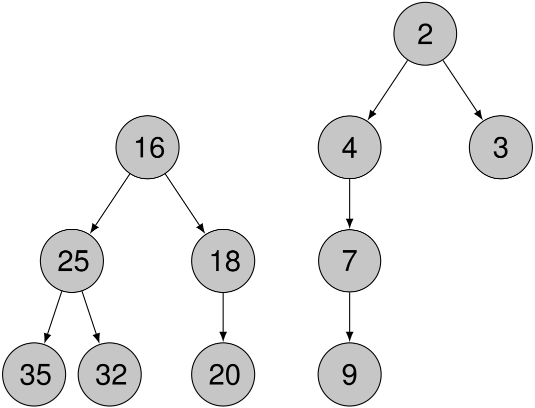


LIS and Hammersley's process

Top of piles in patience sorting = live particles in *Hammersley's process:*

- Particles: random real numbers $X_i \in (0, 1)$.
- Particle X_i kills closest live particle $X_j > X_i$ (if any)

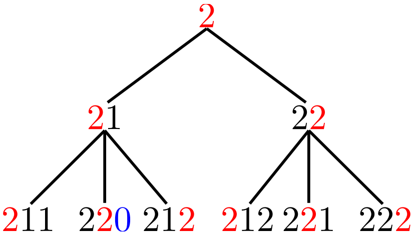
- studied in the area of interacting particle systems
- relative of a more famous process, the so-called Totally Asymmetric Exclusion Process (TASEP)

Aldous-Diaconis: Most illuminating proof of $E[LIS(\pi)] \sim 2\sqrt{n}$, analysis of the so-called hydrodynamic limit of Hammersley's process.

Hammersley's process with k lifelines (HAM_k):

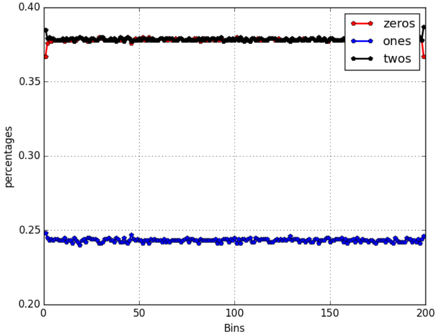
- Particles: slots in patience heaping
- Particles: random $X_i \in (0,1)$, initially k lives.
- X_i removes one lifeline from closest live $X_j > X_i$ (if any)
- Combinatorially, $k = 2$: Words over alphabet $0, 1, 2$.
- Choose a random position. Put there a 2. Remove 1 from the closest nonzero digit to the right (if any).

$E[\Delta(\#$ of heaps$)]$ = 1 + $E[\#$ of trailing zeros of $w]$



A "physicist's explanation" for the dynamics of HAD_k

- *$n \to \infty$: Limit of W_n =* compound Poisson process. $W_n =$ random string of 0,1,2 (densities c_0, c_1, c_2).
- *Assuming well mixing of digits* evolution equations $\to$ prediction on values of c_0, c_1, c_2.



- $c_0 = c_2 \sim \frac{3-\sqrt{5}}{2} \sim 0.381\ldots$, $c_1 \sim \sqrt{5} - 2 \sim 0.236\ldots$
- Distribution of trailing zeros: asymptotically geometric
- From this: $E[\Delta(\#\ heaps.)\ at\ stage\ n] \sim \frac{1+\sqrt{5}}{2} \cdot \frac{1}{(n+1)}$.

How could we (attempt to prove) this ?

- Study the formal power series of HAD_k: $F_k(w)$= multiplicity of word w in the process.
- Obtain probability by dividing by $|w|!$.

Sample Theorem from the paper:

$L_H^k =$ the set of words that satisfy the following condition:

- for all prefixes z of w

$$(*)|z|_k - \sum_{i=0}^{k-2}(k-i-1)\cdot|z|_i > 0.$$

(in particular w starts with a k).

Proof sketch

Direct inclusion: count transitions

- $k \to k + (k-1)$.
- $(k-1) \to k + (k-2)$: $a_{k-1} \geq 0$ moves.
- $\ldots$
- $1 \to k + 0$ $a_1 \geq 0$ moves..
- $\lambda \to k$: $a_0 \geq 1$ moves..
- So $|z|_0 = a_1, |z|_1 = a_2 - a_1, \ldots, |z|_k = a_0 + a_1 + \ldots + a_{k-1}$. Compute a_i in terms of $|z|_j$ and use condition $a_0 > 0$.

Opposite inclusions: several lemmas

- All words in L_H^k start with a k.
- L_H^k closed under prefix.

- All words with $(*) = 1$, $(*) > 0$ in L_H^k

Proof sketch

The induction

- $n = 1$: $z = k$, true.
- $n - 1 \Rightarrow n$. Let z be on the r.h.s. with $|z| = n$.
- Define w to be the word obtained from z by deleting rightmost k and increasing by 1 the next letter.
- w's definition correct: Deleted k not the last letter, otherwise some prefix of z would have $(*) = 0$.
- $|w| = n - 1$. All prefixes of w have $(*) > 0$: any decrease (if any) in the number of k's offset by increase in the value of the next letter.
- By induction $w \in L_H^k$. But w yields z in one step.
- Finally, every word z in the r.h.s. prefix of a word, e.g. $z(k-2)(k-2)\ldots$, with $(*) = 1$.

Algorithm for computing F_k

Input: $k \geq 1, w \in \Sigma_k^*$

$S := 0.\ w = w_1 w_2 \ldots w_n$

if $w \notin L_H^k$ **return** 0

if $w == \text{`}k\text{`}$ **return** 1

for i in 1:n-1

if $w_i == k$ and $w_{i+1} \neq k$

let $r = min\{l \geq 1 : w_{i+l} \neq 0 \text{ or } i + l = n + 1\}$

for j in 1:r-1

let $z = w_1 \ldots w_{i-1} w_{i+1} \ldots w_{i+j-1} 1 w_{i+j+1} \ldots w_{i+r} \ldots w_n$

$S := S + ComputeMultiplicity(k, z)$

if $i + r \neq n + 1$ and $w_{i+r} \neq k$

let $z = w_1 \ldots w_{i-1} w_{i+1} \ldots w_{i+r-1} (w_{i+r} + 1) w_{i+r+1} \ldots w_n$

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Algorithm for computing F_k

if $w_n == k$

  let $Z = w_1 \ldots\ldots w_{n-1}$

  $S := S + ComputeMultiplicity(k, z)$

return S

The constant in the golden-ratio conjecture

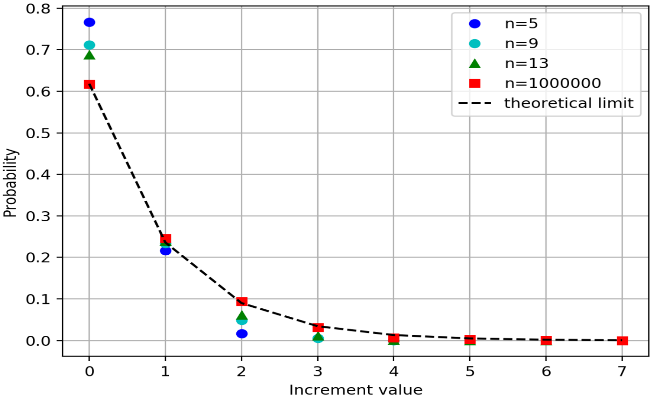

Figure 2: Probability distribution of increments, for $k = 2$, and $n = 5, 9, 13, 1000000$.

Conclusions

Rich problem with many open questions:

- The complexity status of the longest heapable subsequence (Byers et al. 2011)
- The formal power series of the Ham_k process
- The "golden ratio" conjecture (CPM'2015, also manuscript, 2018)
- Heapability of sets/seqs. of random intervals (2018)

$$\lim_{n\to\infty} \frac{E[k\text{-width}(P)]}{n} = \frac{1}{k+1}.$$

- Heapability of random d-dimensional posets (DCFS'2016) (random model: Winkler, Bollobas and Winkler)

$$E[k\text{-width}(P)] = \Theta(\log^{d-1}(n)).$$

Thank you. Questions ?