Interactive Particle Systems and Random Walks on Hypergraphs

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We study hypergraph versions of interacting particle systems and random walks, notably generalizations of coalescing and annihilating random walks. Their definition is motivated by the problem of analyzing the expected running time of a local search procedure for the k-XOR SAT problem, as well as a certain constrained triad dynamics in the theory of social balance.

1 Introduction

Interacting particle systems are discrete dynamical systems, usually defined on lattices, studied intensely in Mathematical Physics [12]. They can be investigated on finite graphs as well [9, 10] as finite Markov chains, and correspond via *duality* to certain types of random walks [1]. The analysis of these models can sometimes be used to bound the mixing time of certain (hyper)graph coloring procedures [10, 5].

A recent development in interacting particle systems and random walks is the extension of the theory to hypergraphs [5, 13, 8, 7, 3] and simplicial complexes [17, 14]. We contribute to this direction by studying hypergraph analogues of coalescing/annihilating random walks and the voter model.

Besides the obvious fundamental interest of such a generalization, the models we consider are motivated by several apparently unrelated applications: the analysis of a local search procedure for the XOR-SAT problem, the theory of social balance [2] and that of lights-out games[16]. On the other hand the study of these systems, though it preserves some properties from the graph case has additional interesting features: for instance for so-called annihilating random walks on hypergraphs the number of particles is **not** in general nondecreasing (as it is in the graph case) and the structure of recurrent states is interestingly constrained by systems of linear equations similar to the ones used to analyze lights-out games [16]. On the other hand, in coalescing random walks on hypergraphs there may be more than one copy of an initial "ball" and the process is naturally described using *multisets* rather than sets of balls.

The plan of the paper is as follows: first we define the models we are interested in and outline their motivation. In Section 3 we present the (still open in general) issue of reachability and recurrence for annihilating random walks, together with a result settling this for our intended applications. In such a setting, our main result (Theorem 4 in Section 4) upper bounds expected annihilation time in terms of a Cheeger-like constant of the hypergraph. We conclude with an application of this result to the analysis of the running time of a RandomWalk algorithm for instances of k-XOR-SAT and other (brief) remarks.

2 Preliminaries

Hypergraphs considered in this paper are *simple*: for every two hyperedges e, f, $|e \cap f| \leq 1$. On the other hand we will allow *self-loops*, i.e. hyperedges e with $|e| = 1$. We will even allow multiple self-loops to

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Algorithm RandomWalk(Φ):

Start with an arbitrary assignment U.
while (there exists some unsatisfied clause)
  pick a random unsatisfied clause C
  change the value of a random variable of C in U
return assignment U.

Figure 1: The RandomWalk algorithm

the same vertex. A *multiset* is a set whose elements have a (positive) multiplicity. The *disjoint union* of multisets A and B, denoted $A \sqcup B$, is the multiset that adds up multiplicities of an element in A and B.

2.1 Motivating examples

In this paper we are concerned with a version of satisfiability called k-XOR-SAT:

Definition 1. *Given constant $k \geq 2$, an instance of k-XOR-SAT is a linear system of equations $A \cdot \overrightarrow{x} = \overrightarrow{b}$ over boolean field $\mathbf{Z}_2$, where A is an $m \times n$ matrix, for some $m,n \geq 1$, $\overrightarrow{x} = (x_1, x_2, \ldots, x_n)^T$ is an $n \times 1$ vector, $\overrightarrow{b} = (b_1, b_2, \ldots, b_m)^T$ is an $m \times 1$ vector, and each equation has exactly k variables.*

Though k-XOR-SAT can be solved in polynomial time by Gaussian elimination, we will analyze instead a local search procedure, the RandomWalk algorithm displayed in Figure 1. The analysis of local procedures is quite complicated in general, so performing such an analysis is, we feel, interesting. Indeed, we will obtain rigorous upper bounds on the expected running time of RandomWalk on solvable instances in terms of measurable parameters of these instances.

A second motivation comes from the physics of complex systems and is given by the following dynamics, first investigated by Antal, Krapivsky and Redner:

Definition 2. Constrained Triadic Dynamics*[2, 11]. We start with a graph $G = (V, E)$ whose edges are labeled $0/1$. A triangle T is G is called* balanced *if the sum of the labels of its edges is 0 (modulo 2). At any step t, we chose an imbalanced triangle T uniformly at random and we change the sign of a random edge of T (thus making T balanced). The move might, however, make other triangles unbalanced.*

CTD can be modeled by the RandomWalk algorithm on an instance of 3-XOR-SAT [15]. As further shown in [11], one can sometimes analyze CTD using duality. We give here a slightly more general version, suitable for the analysis of k-XOR-SAT:

Definition 3. *Given instance Φ of k-XOR-SAT, the* dual *$D(\Phi)$ of Φ is an undirected hypergraph with self-loops $D(\Phi) = (\overline{V}, \overline{E})$ defined as follows: $\overline{V}$ is the set of equations of Φ. Hyperedges in $D(\Phi)$ correspond to variables in Φ and connect all equations containing a given variable. In particular we add a self-loop to an equation (vertex) v if it contains a variable appearing* only in v. *We may even add multiple self-loops to the same vertex.*

Note that if Φ is an instance of $k - XOR - SAT$ then $D(\Phi)$ is a k**-regular hypergraph** (i.e. every vertex has degree exactly k).

2.2 Voter and random walk models on hypergraphs

When viewed by duality the RandomWalk algorithm translates to:

Definition 4. *Let $H = (V,E)$ be a connected hypergraph. Define a* annihilating random walk on H *by the following:* (a). Initial state: *Initially:* $A_i \in \{0,1\}$. *We will call a vertex i with $A_i = 1$* live. (b). Moves: *Choose random pair i, e consisting of* **live node** *i and hyperedge $e = (i, j_1, \dots, j_k)$ containing i. Simultaneously set $A_v = A_v \oplus A_i$ for all $v \in e$ (including $v = i$, which will result in $A_i = 0$).*

It will be, however, more convenient to analyze annihilating random walks on k-uniform hypergraphs with two additional changes:

- (a). first, we will study the *lazy* version of a.r.w., the one in which the choice of node i is not restricted to live nodes only.
- (b). second, we will study annihilating random walks in *continuous*, rather than discrete, time.

Definition 5. *[*Lazy a.r.w. on hypergraphs]:

Let $H = (V,E)$ be a connected k-uniform hypergraph. Define a *annihilating random walk on H* by the following: (a). Initial state: Initially: $A_i \in \{0,1\}$. We will call a vertex i with $A_i = 1$ *live*. (b). Moves: Choose random node i and random edge $(i, j_1, \dots, j_k)$ containing i. Simultaneously set $A_v = A_v \oplus A_i$ for all $v \in e$ (including $v = i$, which will result in $A_i = 0$).

In discrete time the annihilation time of lazy a.r.w. provides, of course, only upper bound on annihilation in a.r.w (and, thus, convergence of the RandomWalk algorithm in the dual model). The introduction of continuous time does not create additional problems by the well-known equivalence between discrete and continuous time Markov processes with independent Poisson clocks.

Next we define an analogue of coalescing random walks to hypergraphs as well:

Definition 6. *[*(Lazy) coalescing random walks (c.r.w.) on hypergraphs]:

Let $H = (V,E)$ be a connected hypergraph. Each vertex holds a multiset of label A_i. Define a *coalescing random walk on H* by the following: (a). Initial state: $A_i \subseteq \{i\}$. Note that $\mathscr{B} := A_1 \cup A_2 \cup \dots \cup A_n \subseteq [n]$. We will call a vertex i with $|A_i| =$ odd *live*. (b). Moves: Updating node i and hyperedge $e = (i, j_1, j_2, \dots, j_k)$ (according to a Poisson clock) proceeds by aking $A_{j_r} := A_{j_r} \uplus A_i$, for $r = 1, \dots, k$ and $A_i = \emptyset$. Here $\uplus$ refers to the **multiset sum**, i.e. union with multiplicities. Note that the move never destroys any label (always $A_1 \cup A_2 \cup \dots \cup A_n = [n]$) but may make some indices i satisfy $|A_i| =$ even. (c). Parity (coalescence) from $\mathscr{B}$: Given set of vertices $\mathscr{B} \subseteq V(G)$, $c_{coal}(H, \mathscr{B})$ is the minimum $t \geq 0$ such that (if starting from configuration with $A_v = v$ exactly when $v \in \mathscr{B}$) at time t it holds that $|A_j| =$ even for every j.

Finally, the "dual" to coalescing random walks, a hypergraph analog of the voter model:

Definition 7. *[*Voter model on hypergraphs]:

Let $H = (V,E)$ be a connected hypergraph. Define a *voter model on H* by the following: (a). Initial state: $A_i = \{i\}$. Note that $A_1 \cup A_2 \cup \dots \cup A_n = [n]$. W (b). Moves: Updating node i and hyperedge $e = (i, j_1, j_2, \dots, j_k)$ (chosen according to an independent Poisson clock) results in setting $A_i = \uplus_{r=1}^{k} A_{j_r}$. Note that the operation may decrease the number of different "opinions" present in the system, if such opinions were only held by node i. (c). Parity of opinions: Given $\mathscr{B} \subseteq V(H)$, *parity time* $c_{VM}(H, \mathscr{B})$ is the minimum time t such that every initial opinion is present an even number of times (perhaps zero) among nodes in $\mathscr{B}$.

2.3 Spectral measures on hypergraphs

The classical definition [4] of the Cheeger constant of a connected graph is:

Definition 8. *The edge expansion (or Cheeger constant) $h(G)$ of graph G is defined as*

$$h(G) = \min_{0<|A|\leq \frac{n}{2}} \frac{|E(A,\overline{A})|}{|A|},$$

where $E(A,\overline{A})$ is the edge boundary of set A*, that is, the set of edges with one endpoint in set A and another one in its complement.*

On the other hand the analysis of voter model on graphs required a different variant related to the conductance of a graph from the theory of rapidly mixing Markov chains, and called in [1] *Cheeger time*:

Definition 9. *For a k-regular hypergraph G define* the Cheeger time τ_G *of G by*

$$\tau_G = \sup_{0<|A|<|V|} \frac{k|A||\overline{A}|}{n \cdot |E(A,\overline{A})|},$$

where $E(A,\overline{A})$ is the set of all edges e crossing the cut $(A,\overline{A})$.

The Cheeger constant/time can be generalized to higher dimensions: for instance Parzachevski et al.[14] gave an extension of the Cheeger constant to simplicial complexes (hence hypergraphs as well). Our desired connection with the voter model on hypergraphs will require, however, a specially tailored extension of the latter quantity.

To motivate our definition consider the case when G is a graph and a partition of $V(G)$ into two parts $V(G) = A \cup \overline{A}$. Each edge e crossing the cut $(A,\overline{A})$ (the set of such edges will be denoted by $E(A,\overline{A})$) maps bijectively to

- A pair (v,e) with $v \in A$ and $e \ni v$ such that $|e \cap A| = 1$. The set of such pairs will be denoted by $E^+(A,\overline{A})$.
- A pair (w,e) with $w \in \overline{A}$ and $e \ni w$ such that $|e \cap A| = 1$. The set of such pairs will be denoted by $E^-(A,\overline{A})$.

Thus $E(A,\overline{A})| = |E^+(A,\overline{A})| = |E^-(A,\overline{A})|$. In particular we can replace $|E(A,\overline{A})|$ in the definition of the Cheeger constant/time by equivalent quantity $min\{|E^+(A,\overline{A})|, |E^-(A,\overline{A})|\}$.

We extend these quantities to hypergraphs in the following (non-obvious) way:

Definition 10. *Given hypergraph H and cut $A,\overline{A}$ define*

- *$E^+(A,\overline{A})$ to be the set of pairs (v,e) with $v \in A$ and $e \ni v$ such that $|e \cap A|$* **is odd.**
- *$E^-(A,\overline{A})$ to be the set of pairs (w,e) with $w \in \overline{A}$ and $e \ni w$ such that $|e \cap A|$* **is odd.**

The reason for these admittedly unusual ”odd cut sizes” will become clear in the proof of Theorem 4.

Definition 11. *For a k-regular hypergraph H define* the Cheeger time τ_H *as*

$$\tau_H = \sup_{0<|A|<|V|} \frac{k|A||\overline{A}|}{n \cdot min\{|E^+(A,\overline{A})|, |E^-(A,\overline{A})|},$$

Note that the previously defined quantity is nontrivial, as it coincides (when H is a graph) with the Cheeger time of H defined in [1].

3 Annihilating random walks: reachability and recurrence

If the hypergraph H is a graph the long-term structure of configurations of the a.r.w. is simple: either a single live site survives (if $|V(H)|$ is odd) or none. In the general case the behavior is more complicated: the number of live nodes is **not** necessarily decreasing, as is the case in the graph setting. There may be, therefore, recurrent states different from $\mathbf{0}$ and those states with a single live node.

The structure of recurrent states is easy to determine, though, for satisfiable instances of k-XOR-SAT:

Theorem 1. *Let Φ be a satisfiable instance of k-XOR-SAT. Let X_1 be an arbitrary assignment and let w_1 be the configuration in the hypergraph $D(\Phi)$ corresponding to X_1. Finally let w_2 be the "all-zeros" configuration. Then w_2 is reachable from w_1. In other words a satisfying assignment X_2 for Φ can be found from initial assignment X_1 by means of moves of the RandomWalk algorithm.*

Proof. Assume that X is a solution to the system $A \cdot x = b$. Let X_0 be an initial assignment. We will prove that a solution of the system is reachable from X_0 by induction on k, the Hamming distance between X_0 and X.

- **Case** $k = 0$**.** Then $X_0 = X$ and there is nothing to prove.
- **Case** $k = 1$**.** Then X_0 and X differ on a single variable z. Let w be an equation containing z. Then X_0 does not satisfy w (as X, which only differs on z, does). Choosing equation w and variable z we reach X from X_0.
- **Case** $k \geq 2$. If there is an equation w not satisfied by X_0 (but satisfied by X) then w must contain a variable on which X_0 and X differ. Let z be such a variable. Then by flipping z one can reach from X_0 an assignment X_1 at Hamming distance $k-1$ from X. Now it is easily seen that system $H(X_1, X)$ has solutions: any solution of $H(X_0, X)$ with the value of z flipped. By the induction hypothesis one can reach a solution from X_1, therefore from X_0.

□

In the general case setting can give [11] a necessary condition for reachability:

Definition 12. *For every pair of boolean configurations $w_1, w_2 : V(H) \to \mathbf{Z}_2$ on hypergraph H we define a system of boolean linear equations $H(w_1, w_2)$ as follows: Define, for each hyperedge e a variable z_e with values in $\mathbf{Z}_2$. For any vertex $v \in V(H)$ we define the equation $\sum_{v \in e} z_e = w_2(v) - w_1(v)$. In the previous equation the difference on the right-hand side is taken in $\mathbf{Z}_2$; also, we allow empty sums on the left side. System $H(w_1, w_2)$ simply consists of all equations, for all $v \in V(H)$.*

Definition 13. *If x is a state on H and l is a hyperedge of H, define $x^{(l)}(v) = 1 + x(v)$, if $v \in l, x(v)$, otherwise.*

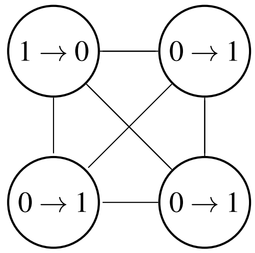
Lemma 1. *If state w_2 is reachable from w_1 then the system of equations $H(w_1, w_2)$ has a solution in $\mathbf{Z}_2$.*

Proof. Let P be a path from w_1 to w_2 and let z_e be the number of times edge e is used on path P (mod 2). Then $(z_e)_{e \in E}$ is a solution of system $H(w_1, w_2)$. Indeed, element $w(v)$ (viewed modulo 2) flips its value any time an edge containing v is scheduled.

□

In [11] we claimed a partial converse of Lemma 1. As the result below shows, though, the converse of Lemma 1 is however **not** true in graphs, or even in hypergraphs with no graph edges:

Lemma 2. *The following are true: There exists (a). a connected graph (i.e. all hyperedges have size two) H, or (b). a connected hypergraph H that contains no graph edges; there also exist two configurations w_1, w_2 on H such that system $H(w_1, w_2)$ has solutions in $\mathbf{Z}_2$, yet w_2 is not reachable in H from w_1.*



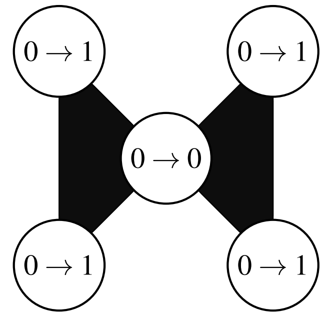


Figure 2: Unreachability in (a). a graph (b). a hypergraph with no graph edges. In each vertex the label of configuration w_1 is written first, that of w_2 next.

Proof.

1. Let H be the complete graph K_4 and let w_1 be 1 at a single vertex v. Let w_2 be the configuration with ones at *every vertex but* v. System $H(w_1, w_2)$ has solution $z_e = 1$ for every edge e, yet w_2 is **not** reachable from w_1, as w_1 has a single one and w_2 has three, but on a graph the number of ones does not increase.

2. Let H be a hypergraph consisting of two triangles T_1, T_2 with a common vertex (Figure 3 (b)). Let w_1, w_2 be the configuration described in that figure: in each node the first value is that of w_1, which should change (represented by $\to$) into that of w_2.

   It is easy to see that system $H(w_1, w_2)$ has a solution z with $z(f) = 1$ for both hyperedges. Yet w_2 is not reachable from w_1. Indeed triangles with three labels of one have no preimage under scheduling of triangles. So the only preimages of state w_2 are itself and the two obtained by reversing one of the triangles.

□

While we raise the complexity of reachability as an open problem, we believe it is possible to ”patch” the result in [11] (perhaps by imposing meaningful restrictions on states w_1, w_2) and further extend it in order to provide a large class of reachability instances for which the necessary condition in Lemma 1 is also sufficient.

4 Upperbounding annihilation on hypergraphs

A particular setting where the previous result is applicable is given by our motivating examples: the XOR-SAT problem if the system has a solution and the CTD for social balance. Therefore with these cases in mind we can define the hypergraph analogue of annihilation time:

Definition 14. Annihilation: *Given set of vertices $\mathscr{B} \subseteq V(G)$, $c_{ann}(G, \mathscr{B})$ is the minimum $t \geq 0$ such that, if we start with balls exactly at vertices of $\mathscr{B}$, at time t we have $A_i = 0$ for all i.*

Definitions 4 and 6 enable upperbounding annihilation on hypergraphs. This is done by extending a coupling argument valid in the case of graphs:

Theorem 2. *Suppose G is a hypergraph without graph partitions and w_1 is a configuration such that the a.r.w. on G can reach annihilation. Then for any $\mathscr{B} \subseteq V(G)$ one can couple the coalescing and annihilating random walks on G such that $c_{ann}(G, \mathscr{B}) \leq c_{coal}(G, \mathscr{B})$.*

Proof. We will define the following stochastic process P:

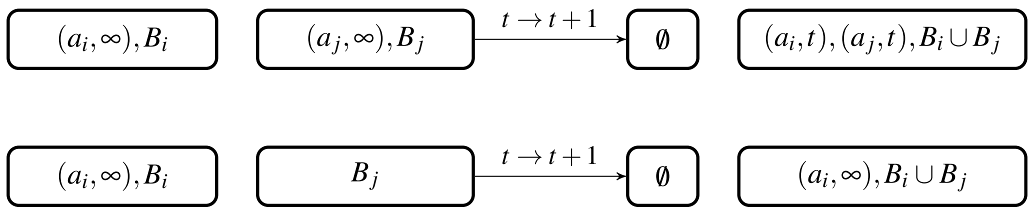


Figure 3: The two cases of stochastic process P. Only two nodes inside a common hyperedge are pictured.

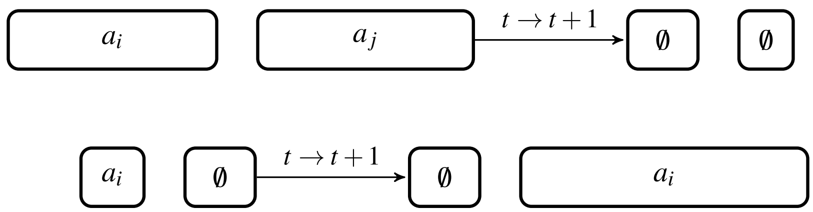


Figure 4: First coupled version: annihilating random graphs (the two cases). Only two nodes inside a common hyperedge are pictured.

1. **Initial state:** $A_i = \{(i,\infty)\}$ for $i \in \mathscr{B}$, $A_i = \emptyset$ otherwise. Note that $A_1 \cup A_2 \cup \ldots \cup A_n = \mathscr{B} \times \{\infty\}$ and that each A_i contains at most one index b_i with $(b_i,\infty) \in A_i$. We will call such a set *live* and b_i *the witness* for A_i. Also denote $B_i = A_i \setminus \{(i,\infty)\}$ if i is live, $B_i = A_i$ otherwise.

2. **Move:** At time t: Choose random vertex i (not necessarily live). Choose random edge $(i, j_1, \ldots, j_k)$. For $r = 1, \ldots, k$

   - If both A_i, A_{j_r} are live then make $A_{j_r} = (B_i \cup B_{j_r}) \cup \{(b_i,t),(b_{j_r},t)\}$.
   - If, on the other hand, at most one of A_i, A_{j_r} is live then make $A_j := A_i \cup A_{j_r}$.

   Finally make $A_i = \emptyset$. Note that if we "move" a dead set A_i to a live set A_j then A_j will still be live.

3. **Stopping:** *Stopping time* $c_P(G)$ is the minimum $t \geq 0$ such that at most one i is live (one if n is odd, none if n is even)

Claim 1. *The following are true:*

1. *P* **observed on** $[n] \times \{\infty\}$ **and moves of live sets only** *is the annihilating random walk on G starting from configuration $\mathscr{B}$. If n is even then at time $c_P(G)$ all particles have annihilated. Consequently $c_{ann}(G,\mathscr{B}) \leq c_P(G,\mathscr{B})$.*

2. *P where we disregard second components in all pairs is identical to the coalescent random walk on G and $c_P(G,\mathscr{B}) = c_{coal}(G,\mathscr{B})$.*

A "proof by picture" is given in Figure 3. There are two cases: j is live or not. In both cases the observed process is identical to the annihilating random walk. Note that if n is even then when coalescence occurs in the c.r.w. all particles have died in the a.r.w.

□

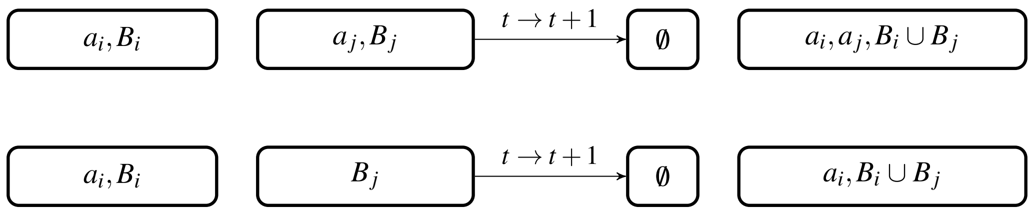


Figure 5: Second coupled version: coalescing random walks (the two cases). Only two nodes inside a common hyperedge are pictured.

The reason a result such as Theorem 2 is interesting is that on graphs (see [1]) $c_{coal}(G)$ is identical (via duality) to coalescence time of voter model $c_{VM}(G)$, which can in turn be upper bounded in terms of a so-called *Cheeger coefficient of graph G*, essentially the inverse of the more well-known Cheeger constant of G. Similar results holds on hypergraphs although we will need to give them in a slightly more general form:

Theorem 3. *For any hypergraph H and any set of vertices $\mathscr{B} \subseteq V(H)$, the coalescence time $c_{coal}(H, \mathscr{B})$ and the parity time of the associated voter model $c_{VM}(H, \mathscr{B})$ are identically distributed.*

Proof. The proof is an adaptation of the classical duality argument [1]: we will define a process on *oriented hyperedges* in H (that is edges with a distinguished vertex) that will be interpreted in two different ways: as parity in the voter model and coalescence in the coalescent random walk.

The process is described in Figure 6. There is a certain difficulty in drawing pointed events in hypergraphs. In the figure we chose (in the interest of readability) not to represent the hyperedges vertically, but as triangles with a spatial extent, instead marking on the time axis the moment the given hyperedge event occurs (times t_1 and t_2 in the coalescing random walk). Horizontal lines (e.g. for ball 3 between moments t_1 and t_2) refer to histories not interrupted by any hyperedge event between the corresponding times.A horizontal line may be interrupted by a hyperedge event pointed at the given node.

A *left-right path P* between node i and node j is a sequence of hyperedge events and horizontal lines such that:

- P starts with a horizontal line of node i and ends with a horizontal line of node j.
- Every horizontal line of a node is followed by a hyperedge event with the corresponding node being pointed.
- Every hyperedge event is followed by an unique horizontal line corresponding to a **non-pointed node**.

For instance, in the picture from Figure 6 we have represented three left-right paths, between node 2 and each of nodes 1,4,5.

In the c.r.w. the activation of a hyperedge $e = [j \to i_1, i_2, \dots i_r]$ pointed at vertex j is interpreted as vertex j being chosen (together with edge e), thus sending a copy of its cluster of balls to all other neighbors.

In the voter model the activation of a hyperedge $e = [j \to i_1, i_2, \dots i_r]$ pointed at vertex j is interpreted as j adopting the multiset union of opinions of $i_1, i_2, \dots, i_r$.

For instance, in the picture in Figure 6:

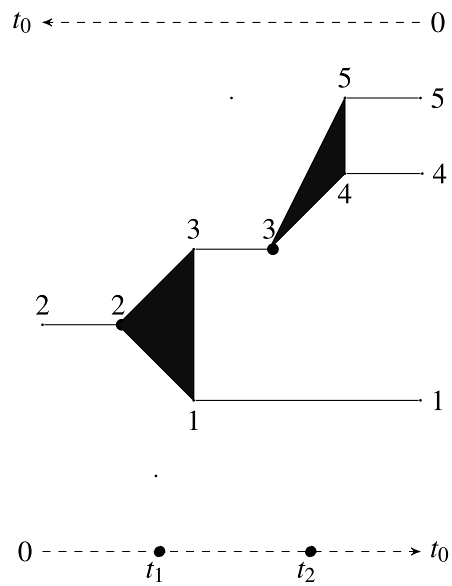


Figure 6: Coupling the coalescing random walk and the voter model. Time runs from left to right in the coalescing random walk and right to left in the voter model. At time t_1 (in the c.r.w.) copies of balls at (pointed) node 2 are sent to nodes 1 and 3. Similarly, at time t_2 copies of cluster at (pointed) node 3 are sent to nodes 4 and 5.

- in the c.r.w., assuming that initially $A_i = \{i\}$, $i = 1,5$, at moment t_0 we have $A_1 = \{1,2\}$, $A_2 = \emptyset$, $A_3 = \emptyset$, $A_4 = \{2,3,4\}$, $A_5 = \{2,3,5\}$.
- in the voter model at moment t_0 we have $A_1 = \{1\}$, $A_2 = \{1,4,5\}$, $A_3 = \{4,5\}$, $A_4 = \{4\}$, $A_5 = \{5\}$. Label 3 has disappeared from the system.

Just as in the ordinary c.r.w./voter model, the existence of a left-right path between nodes i and j (e.g. $(2,1),(2,4),(2,5)$) is interpreted as the event:

- In the c.r.w.: "at time t_0 node j holds a ball with label i."
- In the voter model: "at time t_0 node i holds opinion j with multiplicity at least one."

Moreover one path may contribute (when it does) with *exactly one ball/opinion* of a given type.

Consider now the event: "at t_0 every node in $\mathscr{B}$ on the right-hand side is connected to nodes on the left-hand side by an even number of paths".

- In the coalescing random walk this is equivalent to "at t_0 we have parity from $\mathscr{B}$"
- In the voter model this is equivalent to "at t_0 we have parity of opinions on $\mathscr{B}$"

□

The following is our main result. It is, of course, useful only in hypergraphs for which $E^{\pm}(A,\overline{A}) \neq \emptyset$ for all $0 < |A| < |V|$. In that case:

Theorem 4. *Given connected hypergraph H and set $\mathscr{B} \subseteq V(H)$ we have*

$$E[c_{VM}(H,\mathscr{B})] \leq 2n\tau_H \cdot \ln(2).$$

Consequently a similar upper bound holds for the annihilating random walk on H.

Proof. We will sketch the proof for the case where $\mathscr{B} = V(H)$, but the general argument is completely similar.

Consider a partition of the vertices of H (i.e. initial opinions in the voter model) into two parts, D and $\overline{D}$ and consider the following process, similar to the "two party voter model" from [1]:

- At time $t = 0$ start the process with 0 on labels of vertices of $\overline{D}$ ("reds") and 1 on vertices of D ("blues"). In other words opinions in D are counted, whereas opinions in $\overline{D}$ are not.
- each pair v, e composed of a vertex v and an edge $e \ni v$ has an independent Poisson clock. When this clock rings we perform an update by setting

$$A_v = \sum_{w \neq v \in e} A_w \pmod 2.$$

- We denote by D_t the set of vertices labeled 1 at time t, by N_t^D the cardinal of D_t, and by Δ_t^D the difference in the number of ones as a result of the (possible) jump at time $t + dt$.
- Denote by C^D the smallest time $t \geq 0$ when $N_t^D \in \{0, n\}$, where n is the number of vertices of H.

The "two-party model" above can be coupled with the (regular) voter model by interpreting the state $s_t[v]$ of vertex v at time t as a 0/1 value corresponding to the parity of the number of labels of v at time t that are in D (with multiplicities taken into account).

Lemma 3. *We have*

$$Prob[\Delta N_t^D = 1] \geq \frac{1}{\tau_H} \cdot \frac{N_t^D \cdot (n - N_t^D)}{n}$$

and

$$Prob[\Delta N_t^D = -1] \geq \frac{1}{\tau_H} \cdot \frac{N_t^D \cdot (n - N_t^D)}{n}.$$

Proof. N_t^D decreases by one exactly when the chosen vertex v has label 0 and the edge $e \ni v$ contains an odd number of nodes with label 1. Similarly, N_t^D increases by one precisely when the chosen vertex v has label 1 and the edge $e \ni v$ has an odd number of nodes (including v !) with label 1.

The number of distinct vertex-edge pairs in the two-party voter model is precisely kn, since every vertex of H has degree exactly k.

The number of vertex-edge pairs that lead to an increase by 1 is nothing but $|E^+(D_t, \overline{D_t})|$, with E^+ having the meaning from Definition 10. From the definition of τ_H, conditional on the current configuration we have

$$|E^+(D_t, \overline{D_t})| \geq \frac{kN_t^D(n - N_t^D)}{n\tau_H}$$

□

Now choose D uniformly at random: that is, each node $v \in V$ is independently selected into D with probability 1/2 Denote by $\mathbf{D}$ the corresponding random model and $E[C^{\mathbf{D}}]$ the corresponding convergence time.

Suppose at time t we **do not** have coalescence. Let C_t be the resulting configuration. There must be two different nodes v_1, v_2 in C_t which have sets of labels of different parities. Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ the vectors of label parities of vertices v_1, v_2. By our assumption

$$\sum_{i=1}^{n} a_i \neq \sum_{i=1}^{n} b_i \pmod 2.$$

We next apply the following trivial lemma, which simply follows from the fact that the two vectors are distinct and each entry on which they differ is independently chosen in **D** with probability 1/2:

Lemma 4. *Conditional on being in state* C_t*,*

$$Prob[\sum_{j\in \mathbf{D}} a_j \neq \sum_{j\in \mathbf{D}} b_j \ (mod\ 2)] = 1/2.$$

As a consequence we infer, similarly to the graph case [1], that

$$Prob[C^{\mathbf{D}} > t] \geq \frac{1}{2} Prob[C > t].$$

and, finally

$$E[C] \leq 2 \cdot E[C^{\mathbf{D}}] \leq 2 \cdot max_D E[C^D].$$

Now by an analysis similar to the case of voter models on graphs [9] (or, formally, by applying Lemma 10 in [1]) we get

$$E[C^D] \leq n\tau_H \cdot \ln(2)$$

completing the upper bound.

□

4.1 Application to k-XOR-SAT

Putting the last three inequalities together, applying them to k-XOR-SAT and getting back from a continuous to an equivalent discrete time model we get an upper bound on convergence time of *RandomWalk* on solvable instances H of k-XOR-SAT whose dual $D(H)$ is a simple hypergraph:

Theorem 5. *The following holds:*

$$E[RandomWalk] \leq 2m^2 \tau_{D(H)} \cdot \ln(2),$$

where m *is the number of equations in* H.

5 Conclusions and Acknowledgment

The work can be completed in many ways. Complete details and and many more results (for instance upper bounds on annihilation similar to those in [6]) should be a subject for the journal-length version of this paper.

Finally, it would be interesting to see if the running time of other local search procedures, perhaps for more interesting problems like k-SAT can be analyzed in terms of (suitably defined) ”particle systems”.

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