Stochastic Stability in Schelling's Segregation Model with Markovian Asynchronous Update

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Acks

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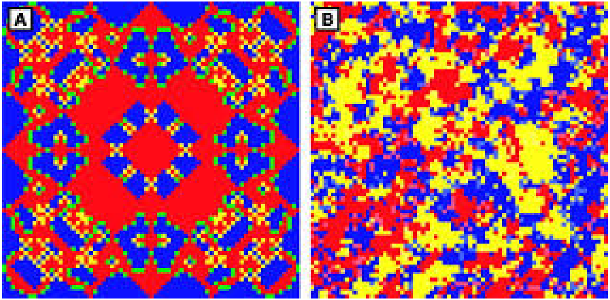
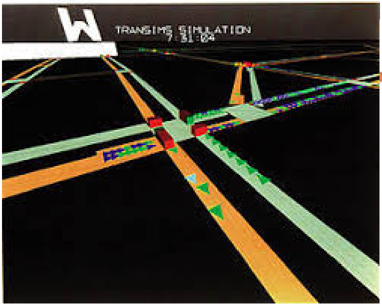
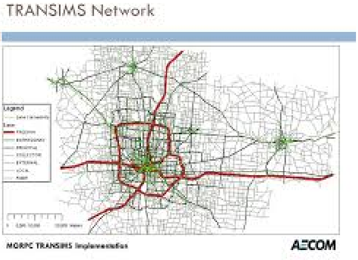
Motivation: update rule matters !



- Nowak & May (Nature 1992): Spatial Prisoners' Dilemma: complex patterns.
- Huberman & Glance (PNAS 1993): this complexity not seen for asynchronous update.

Verification and Validation of Evolutionary Game Models & Social Simulations





- Soc. simulations (e.g. TRANSIMS): increasingly important.
- (When not parallel) many models employ random update.

Is there a single instance when true random asynchronous activation is socially plausible ? Are results crucially dependent on this assumption ?

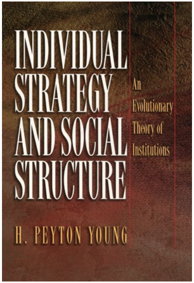
Adversarial Scheduling Analysis of Evolutionary Game Models & Social Simulations

- Adversarial scheduling: (G.I., Marathe, Ravi MSCS'12) Vary scheduler (adversarially), keeping everything else the same. Attempt to infer conditions on the scheduler that cause the baseline result to break/extend.
- This paper: do this for a version of Schelling's Segregation Model.
- Framework: stochastic stability in evol. game theory. Peyton Young (1-D), Zhang, Pollicott& Weiss (2-D).

Take-home message:

If scheduling is nonadaptive (next pair does not depend on system state), then result valid under random scheduling extends. Adaptive schedulers may break this.

Stochastic Stability: Intuition



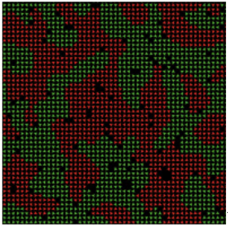
- Best-response update: multiple equilibria (fixed points), actual output path dependent.
- Intuition: adding small amounts of noise can often "choose" one of the equilibria
- Equilibrium selection for risk-dominant equilibria. Emergence of standards/norms: driving on the left/right, gold vs. silver, etc.

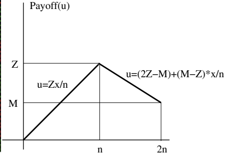
Stochastic Stability: Definitions

- **Definition:** Consider a Markov process P^0 defined on a finite state space Ω. For each $\epsilon > 0$, define a Markov process P^ϵ on Ω. P^ϵ is a regular perturbed Markov process if all of the following conditions hold.
- P^ϵ is irreducible for every $\epsilon > 0$.
- For every $x, y \in \Omega$, $\lim_{\epsilon > 0} P^\epsilon_{xy} = P^0_{xy}$.
- If $P_{xy} > 0$ then there exists $r(m) > 0$, the *resistance of transition $m = (x \to y)$*, such that as $\epsilon \to 0$, $P^\epsilon_{xy} = \Theta(\epsilon^{r(m)})$.

Let μ^ϵ be the (unique) stationary distribution of P^ϵ. A state S is *a stochastically stable strategy* if $\underline{\lim}_{\epsilon \to 0} \mu^\epsilon(S) > 0$.

Schelling's Segregation Model: Our Version





- $N \times N$ rectangular grid with periodic boundary conditions.
- Fields occupied by red/green agents.
- Agents' utility: $u_i(\cdot) = rw(\cdot) + \epsilon$, where $r > 0$, and $w(x)$ is a (weighted) combination of the number of neighbors of x having the same color and the number of neighbors of x having the opposite color.

Scheduling Model

- Random Scheduler: two random agents get picked. If they can improve payoffs they switch. Else:

$$Pr[\text{switch}] = \frac{e^{\beta[u_1(\cdot|\text{switch})+u_2(\cdot|\text{switch})]}}{e^{\beta[u_1(\cdot|\text{switch})+u_2(\cdot|\text{switch})]}+e^{\beta[u_1(\cdot|\text{not switch})+u_2(\cdot|\text{not switch})]}},$$

- 1-D: Peyton Young. 2-D: Zhang (JEBO, 2003).

BASELINE RESULT: Under random scheduling *stochastically stable states* are *maximally segregated*, i.e. maximize a potential function (measuring segregation).

- Markovian asynchronous update: To each pair of vertices e associate p.d. D_e on $V \times V$. If t_i is the pair scheduled at stage i choose t_{i+1}, by sampling from D_{t_i}. $e \in supp(D_e)$.
- Weakly reversible: ($Pr[e \to e'] > 0 \Rightarrow Pr[e' \to e] > 0$.

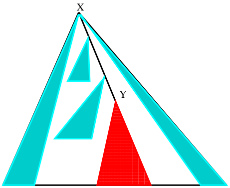
Segregation under non-adaptive scheduling

Theorem:

Under Markovian asynchronous update the stochastically stable states are in the set $\{(s, e) : s \text{ is } \textbf{maximally segregated} \text{ and } e \in E\}$.

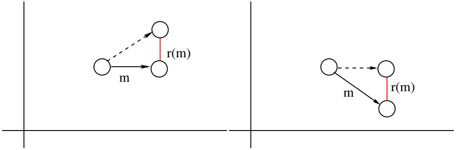
- Simplest form of **nonadaptive scheduling:** next scheduled edge based on last active edge **but not the state/outcome of the last move**.
- Scheduled edge can depend on state (outcome last move): scheduler can (easily) forever preclude segregation.

Proof idea (cheating)



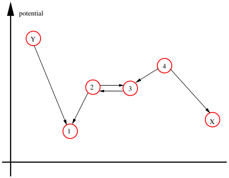
- Dynamics driven by "potential function."
- Use Foster-Young criterion for stochastic stability.
- Tree of states rooted at state j: set T of edges s.t. for any state $w \neq j$ there exists a unique (directed) path from w to j. Resistance of a rooted tree T: sum of resistances of all transitions in T.
- Transform any tree rooted at a non-maximally segregated state into a tree of lower resistance.
- "Reverse" path from X to Y. Transform subtrees of T.

Proof idea (cheating, II)



- Crucial: **connection between potential function and resistance**.
- Resistance $r(m)$ of a move $m = (a_1, j_1) \rightarrow (a_2, j_2)$ only depends on the potential values at three points: a_1, a_2 and a_3 (where a_3 is the state obtained by making the opposite choice)

Proof idea (cheating, III)



- Compare resistances of moves on direct vs reverse path.
- Moves that don't change state: same resistance in both directions
- Other: difference in resistances = change in potential
- Difference in sum of resistances $\equiv$ Difference in potentials between endpoints !

Conclusion:

Maximally segregated: states: highest potential. Always lead to best trees

How am I cheating ?

- Technical difficulty: Markov chain two components: state and last scheduled edge.
- Cannot truly reverse path because second component.
- But: potential of state (s, e) does not depend on e !
- "Reverse": only reverse first component (create new path with reversed projection), add zero-resistance moves to attach trees to new path, etc.)

All of this works

The things I am cheating about are mere technicalities.

Conclusions & Further Work

- Are all maximally segregated states stochastically stable ? Open Question for P&W model.
- (Somewhat) Parallel update ? Peyton Young model:. Auletta et al. (SPAA'2011).
- More general scheduling ? "Influence model"
- How does convergence time relates to network structure ?
- Random Scheduling: Convergence time linear on so-called "close-knit graphs". Does not extend to Markovian contagion: *line graph* L_{2n+1} on $2n+1$ nodes labeled $-n, \ldots, -1, 0, 1 \ldots n$. Random walk from the origin. Convergence time $\theta(n^2)$.

More philosophical ruminations

What about social simulations ?

Can we apply such an analysis not only to mathematical models ?

- Our models/simulation produce stylized facts.
- Some stylized facts more robust, some very brittle.
- Model may display (or lack) "phase transitions" across parameters in their stylized properties.
- In mathematical models: causality easi(er) to identify.

Need a logic/discipline of stylized facts in modeling !

Thank you. Questions ?