Vector partitions, multi-dimensional Faà di Bruno formulae and generating algorithms☆

Flavius TURCU*, Cosmin BONCHIŞ**,***, Mohamed NAJIM*

** IMS - Bordeaux, Equipe Signal et Image, France*
*** Research Institute e-Austria Timişoara, România*
**** West University of Timişoara, România*

Abstract

In the paper we introduce and study partitions of vectors in $\mathbb{N}^p$, as a natural extension of the classical partitions of integers. We show that these vector partitions are the suitable combinatorial notion in extending the Faà di Bruno formula to a general multi-variable setting, as well as in generalizing Bell polynomials. The Adomian polynomials can be obtained from this framework as a particular case. We also give a recursive algorithm which is proved to generate all the vector partitions without repetitions, and which can be used in numerical applications of extended Faà di Bruno formulae and generalized Bell polynomials.

Key words: vector partitions, multi-dimensional Faà di Bruno formulae, multi-dimensional Bell polynomials

1. Introduction

Many problems and numerical applications require expressing the composition $h = g \circ f$ of two power series f and g as a power series and computing its coefficients from the coefficients of f and g. When Taylor series are involved, an alternative formulation regards the computation of n^{th} order derivatives of $g \circ f$ from the derivatives of f and g up to the order n.

Identifying coefficients in a composition of power series is basically a combinatorial approach to rearrange and group terms arising in multinomial expansions. This rearrangement is naturally connected to the set of the *partitions* of an integer n, i.e. all the distinct possibilities of writing n as a sum of positive integers, the order of the terms left aside. Formally a partition of n can be regarded as a function $\pi : \mathbb{N}^* \to \mathbb{N}$ satisfying $\sum_{j\in\mathbb{N}^*} j\pi(j) = n$, each value $\pi(j)$ indicating how many times the number j appears in the partition. We will de-

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note by $|\pi| = \sum_j \pi(j)$ the *cardinal* of the partition π, representing the number of terms in the sum. We will also use the notation $\pi! = \prod_j \pi(j)!$.

In the case when f and g are functions of one variable, the n^{th} order derivative of $g \circ f$ is given by the well known Faà di Bruno formula

$$\frac{d^n}{dz^n} g(f(z)) = \sum_{\pi \in P_n} \frac{n!}{\pi!} g^{(|\pi|)}(f(z)) \prod_j \left(\frac{f^{(j)}(z)}{j!} \right)^{\pi(j)} \tag{1}$$

where P_n denotes the set of all the partitions of n. The Faà di Bruno formula is often written in a form that emphasises the Bell polynomials, by grouping together in (1) the partitions of cardinal $k = 1, 2, ...$:

$$\frac{d^n}{dz^n} g(f(z)) = \sum_{k=1}^{n} g^{(k)}(f(z)) B_{n,k}(f'(z), f"(z), \ldots, f^{(n-k+1)}(z)), \tag{2}$$

where the Bell polynomials $B_{n,k}$ for $1 \leq k \leq n$ are defined by

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\substack{\pi \in P_n \\ |\pi| = k}} \frac{n!}{\pi!} \prod_j \left(\frac{x_j}{j!} \right)^{\pi(j)} \tag{3}$$

In a multi-dimensional framework, a context that has been widely investigated is $f : \mathbb{R} \to \mathbb{R}^q$ and $g : \mathbb{R}^q \to \mathbb{R}$. The interest for this setting is motivated by the Adomian decomposition method for numerically solving differential equations [2], where the coefficients of the composed power series are regarded as Adomian polynomials. In this type of Faà di Bruno extension, the role of the classical partitions is taken by what one could call *vector-valued* partitions of an integer n, i.e. functions $\pi = [\pi_1, \ldots, \pi_q] : \mathbb{N}^* \to \mathbb{N}^q$ satisfying $\sum_{l=1}^{q} \sum_j j\pi_l(j) = n$, each such function giving rise to a term in an Adomian polynomial. These extended partitions implicitly appear under different forms in various papers such as [1] or [3] (in this latter paper they are viewed as *index matrices*), where different algorithms to generate them are given.

In this paper we deal with the general case $f : \mathbb{R}^p \to \mathbb{R}^q$ and $g : \mathbb{R}^q \to \mathbb{R}$.

In contrast to the case $p = 1, q > 1$ considered in Adomian decompositions, our initial interest was rather to the opposite framework $p > 1, q = 1$. This setting appears in numerical applications in the framework of multi-dimensional filter design, based on the theory of multi-variable Hardy spaces. One typical filter design problem requires the computation of analytic functions with a given absolute value on the distinguished boundary of the unit (poly)disk ([5],[6]). Special solutions to this problem are the so called outer functions, which can be expressed as exponentials of Poisson-kernel integrals (see e.g. [4]). As such, their numerical estimation means computing the coefficients of $g \circ f$, where f is a p-variable power series (representing a Poisson integral), and $g(z) = e^z$.

This particular case $p > 1, q = 1$ will be first considered in Section 2, not only because of its own motivation, but mainly because many aspects of the general case $p > 1, q > 1$ are either related or reducible to this case. We show that, in contrast to the case $p = 1, q > 1$ of Adomian polynomials, featuring vector-valued partitions of integers, the context here naturally produces the opposite, i.e. integer-valued partitions of vectors.

We define a partition of a vector $\alpha \in \mathbb{N}^p \setminus \{\theta\}$ in exactly the same manner as for integers, as being any way to write α as a sum of vectors in $\mathbb{N}^p \setminus \{\theta\}$, the order of the terms left aside (the notation θ stands for the origin of $\mathbb{N}^p$). When regarded as a function, a partition of a vector α is a map $\pi : \mathbb{N}^p \setminus \{\theta\} \to \mathbb{N}$ satisfying $\sum_\beta \beta\pi(\beta) = \alpha$.

By the means of vector partitions we give the generalized Faà di Bruno formula for a composition $\mathbb{R}^p \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$, in both power series and derivative versions. We also give their naturally related extension of the Bell polynomials, after showing some properties of certain classes of vector partitions.

Section 3 provides the generalizations of the Faà di Bruno formulae and of the Bell polynomials in the general case $p > 1, q > 1$. Here the proper notion of partition is a combination of the two extensions mentioned before. More precisely, we call a q-valued partition of a vector α in $\mathbb{N}^p$ a function $\pi = [\pi_1, \ldots, \pi_q] : \mathbb{N}^p \setminus \{\theta\} \to \mathbb{N}^q$ satisfying $\sum_\beta \beta \sum_{j=1}^q \pi_j(\beta) = \alpha$. We also show some properties of certain classes of q-valued partitions of vectors and describe how they can be constructed from single-valued partitions of vectors.

Finally Section 4 is dedicated to the algorithmic construction of vector partitions classes, which can be used in numerical applications of the Faà di Bruno extensions described in the previous sections. We give here a recursive algorithm to generate all the partitions of a vector from the partition set of one of its predecessors in either coordinate. We prove that the algorithm produces all the successor's partitions without repetitions. The idea of the algorithm is a non-trivial generalization of a standard algorithm for generating number partitions.

2. Partitions of vectors and Faà di Bruno extensions

Let us first consider the case of a p-variable power series, denoted $f(\mathbf{z})$ with $\mathbf{z} = (z_1, \ldots, z_p)$, and a one variable power series $g(w)$. Without restraining the generality we may suppose they are both centered at the origin:

$$f(\mathbf{z}) = \sum_{\alpha \in \mathbb{N}^p} a_\alpha \mathbf{z}^\alpha \qquad g(w) = \sum_{n \in \mathbb{N}} b_n w^n, \tag{4}$$

with the standard multi-index notation $\mathbf{z}^\alpha = z_1^{\alpha_1} \cdots z_p^{\alpha_p}$.

Suppose the composition $h(\mathbf{z}) = g(f(\mathbf{z}))$ is written as the power series

$$h(\mathbf{z}) = \sum_{\alpha \in \mathbb{N}^p} c_\alpha \mathbf{z}^\alpha \tag{5}$$

whose coefficients $(c_\alpha)_{\alpha\in\mathbb{N}^p}$ need to be computed from $(a_\alpha)_{\alpha\in\mathbb{N}^p}$ and $(b_n)_{n\in\mathbb{N}}$.

In order to do this, fix an integer n and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_p)$ in $\mathbb{N}^p \setminus \{\theta\}$ and let us look at terms in $\mathbf{z}^\alpha$ in the multinomial expansion of

$$\underbrace{(\sum_{\beta\in\mathbb{N}^p} a_\beta \mathbf{z}^\beta)(\sum_{\beta\in\mathbb{N}^p} a_\beta \mathbf{z}^\beta)\cdots(\sum_{\beta\in\mathbb{N}^p} a_\beta \mathbf{z}^\beta)}_{n}. \tag{6}$$

Obviously any such term derives from all possibilities of writing the multi-degree α as a sum of at most n non-zero multi-degrees, and all the possibilities to affect these multi-degrees to the n parentheses.

This naturally implies the partition of the vector α into vectors of non-negative integers, suggesting the following:

Definition 1. *Given a vector $\alpha \in \mathbb{N}^p\setminus\{\theta\}$, a partition of α is any way of writing α as a sum of vectors in $\mathbb{N}^p\setminus\{\theta\}$, the order of the terms left aside. Alternatively, a partition of α is a map $\pi : \mathbb{N}^p \setminus \{\theta\} \to \mathbb{N}$ satisfying $\sum_\beta \beta\pi(\beta) = \alpha$. In this sense we also admit the trivial map $\pi \equiv 0$ as the unique partition of θ.*

As in the one-dimensional case, the number $\pi(\beta)$ indicates how many times the vector β appears in the partition. In particular the support $supp(\pi) = \{\beta : \pi(\beta) \neq 0\}$ indicates which vectors form the partition. Clearly $supp(\pi)$ is a finite set contained in the hyper-rectangular region

$$D_\alpha = \{\beta \in \mathbb{N}^p : \beta_j \leq \alpha_j, 1 \leq j \leq p\}. \tag{7}$$

We will denote P_α the set of all the partitions of α. For $\pi \in P_\alpha$ we denote by $|\pi| = \sum_\beta \pi(\beta)$ the *cardinal* of π, representing the number of terms in the partition π. We also write $\pi! = \prod_\beta \pi(\beta)!$.

Let us go back to identifying terms in $\mathbf{z}^\alpha$ in (6), and fix one partition $\pi \in P_\alpha$ such that $|\pi| \leq n$. Such a fixed partition π indicates that, for each β in the support of π, the term in $\mathbf{z}^\beta$ will be chosen from a number of $\pi(\beta)$ parenthesis. This means a total number of $|\pi|$ parentheses, while the remaining $n - |\pi|$ parenthesis will provide the free term in the series of f. Now the number of possibilities of choosing the parentheses this way obviously coincides with the multinomial coefficient $n!/\Big((n-|\pi|)!\prod_\beta \pi(\beta)!\Big)$.

Therefore, by summing up over all the partitions of α with cardinal not exceeding n, the coefficient of $\mathbf{z}^\alpha$ in (6) is

$$\sum_{\substack{\pi \in P_\alpha \\ |\pi| \leq n}} \frac{n!}{\pi!(n-|\pi|)!}\, a_\theta^{n-|\pi|} \prod_\beta a_\beta^{\pi(\beta)}.$$

This leads to the following multivariable power series version of the Faà di Bruno formula:

Theorem 2.1. *Let $f(\mathbf{z}) = \sum_{\alpha\in\mathbb{N}^p} a_\alpha \mathbf{z}^\alpha$ and $g(w) = \sum_{n\in\mathbb{N}} b_n w^n$. The coefficients of the composed power series $h(\mathbf{z}) = g(f(\mathbf{z}))$ are*

$$c_\alpha = \sum_{\pi\in P_\alpha} \frac{\prod_\beta a_\beta^{\pi(\beta)}}{\pi!} \left(\sum_{n\geq|\pi|} b_n n! \frac{a_\theta^{n-|\pi|}}{(n-|\pi|)!} \right) \qquad (\alpha\in\mathbb{N}^p). \tag{8}$$

This formula is particularly simple when $g(w) = e^w$, as the sum over n in (8) always equals e^{a_θ} :

Corollary 2.1. *If $f(\mathbf{z}) = \sum_{\alpha\in\mathbb{N}^p} a_\alpha \mathbf{z}^\alpha$ then*

$$e^{f(\mathbf{z})} = e^{a_\theta} \sum_{\alpha\in\mathbb{N}^p} \left(\sum_{\pi\in P_\alpha} \frac{\prod_\beta a_\beta^{\pi(\beta)}}{\pi!} \right) \mathbf{z}^\alpha. \tag{9}$$

The following Faà di Bruno version for partial derivatives can be easily obtained with the same arguments as before, by considering partial Taylor series and a remainder, instead of the whole power series. This version corresponds in fact to the particular case of (8) when $a_\theta = 0$ (which amounts to saying that the series g is centered at $f(\theta)$). In this case the sum over n in (8) is reduced to its first term.

Theorem 2.2. *Let $\alpha\in\mathbb{N}^p$, $f(\mathbf{z})$ a function in p variables that admits partial derivatives up to the multi-order α at $\mathbf{z}_0$ and $g(w)$ a one-variable function $|\alpha|$ times differentiable at $f(\mathbf{z}_0)$. Then*

$$\left(\frac{\partial}{\partial\mathbf{z}}\right)^\alpha g(f(\mathbf{z}_0)) = \sum_{\pi\in P_\alpha} \frac{\alpha!}{\pi!} g^{(|\pi|)}(f(\mathbf{z}_0)) \prod_\beta \left(\frac{\left(\frac{\partial}{\partial\mathbf{z}}\right)^\beta f(\mathbf{z})}{\beta!} \right)^{\pi(\beta)}, \tag{10}$$

with the multi-index notations

$$\alpha! = \prod \alpha_j!, \quad |\alpha| = \sum \alpha_j, \quad \left(\frac{\partial}{\partial\mathbf{z}}\right)^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}\cdots\partial z_p^{\alpha_p}}$$

.

A generalized Bell polynomial form of the Faà di Bruno formula can be derived, exactly as in the one-variable case, by simply regrouping in (10) the partitions of cardinal $k = 1, 2, \ldots$. The only point that needs to be clarified is which are the variables of the extended Bell polynomials.

Let us have a look at this matter in the one-dimensional case which was briefly recalled the introduction. Denote $P_{n,k}$ the set of the partitions of n with cardinal k, each such partition giving rise to a term of the Bell polynomial $B_{n,k}$. It is clear from (3) that each variable x_j effectively appears only in those terms that arise from partitions $\pi\in P_{n,k}$ for which $\pi(j)\neq 0$. Therefore the set of the indexes j that appear in at least one term coincides with the set

$$D_{n,k} := \bigcup_{\pi\in P_{n,k}} supp(\pi). \tag{11}$$

One can easily check that $D_{n,k} = \{1, 2, \dots, n-k+1\}$, which explains why $B_{n,k}$ is defined as a polynomial in $n-k+1$ variables.

Now returning to the multi-dimensional setting, let us denote $P_{\alpha,k}$ the set of the partitions of α with cardinal k. In order to write the generalized Bell polynomials for the Faà di Bruno extension (10), clearly one should first have a look at the set

$$D_{\alpha,k} := \bigcup_{\pi \in P_{\alpha,k}} supp(\pi), \tag{12}$$

which represents, as in the one-dimensional case, the index domain for the variables of the extended Bell polynomial. In order to give an explicit description of this set we will use a couple of notations and one lemma.

For β in $\mathbb{N}^p$ we denote by δ_β the Dirac function at β, whose values are 1 at β and zero elsewhere on $\mathbb{N}^p$. This allows writing partitions $\pi \in P_\alpha$ as linear combinations $\pi = \sum_\beta \pi(\beta)\delta_\beta$. We also denote by $e_1, ..., e_p$ the canonical base in $\mathbb{R}^p$.

Lemma 2.1. *Let α in $\mathbb{N}^p \setminus \{\theta\}$.*

a) For any partition $\pi \in P_\alpha$, $|\pi| \leq |\alpha|$.

b) For any integer $k \in \{1, 2, \dots, |\alpha|\}$ there is a at least one partition $\pi \in P_\alpha$ with $|\pi| = k$.

Proof. To see a) observe that

$$|\alpha| = |\sum_\beta \pi(\beta)\beta| = \sum_\beta \pi(\beta)|\beta| \geq \sum_\beta \pi(\beta) = |\pi|.$$

Note that equality above holds only when $\pi = \sum_{j=1}^p \alpha_j \delta_{e_j}$, whose support contains only vectors in the canonical base.

For b) suppose $k \leq |\alpha|$ and let β be any vector in the hyper-rectangle D_α defined in (7) such that $|\beta| = |\alpha| - k + 1$. Put $\pi = \delta_\beta + \sum_{j=1}^p (\alpha_j - \beta_j)\delta_{e_j}$. Then π is a partition of α, since

$$\sum_\gamma \pi(\gamma)\gamma = \beta + \sum_{j=1}^p (\alpha_j - \beta_j)e_j = \beta + (\alpha - \beta) = \alpha.$$

Moreover

$$|\pi| = \sum_\gamma \pi(\gamma) = 1 + \sum_{j=1}^p (\alpha_j - \beta_j) = 1 + (|\alpha| - |\beta|) = 1 + (k-1) = k,$$

and the proof is complete. $\square$

We can now give the explicit description of the index domain for the variables of the extended Bell polynomials.

Proposition 1. *Let $\alpha \in \mathbb{N}^p \setminus \{\theta\}$ and $1 \leq k \leq |\alpha|$. Then.*

$$D_{\alpha,k} = \{\beta \in D_\alpha : |\beta| \leq |\alpha| - k + 1\}. \tag{13}$$

Proof. For the direct inclusion, suppose β in D_α and π in $P_{\alpha,k}$ such that $\pi(\beta) \neq 0$. Then

$$|\alpha| = \sum_{\gamma} \pi(\gamma)|\gamma| = |\beta| + (\pi(\beta) - 1)|\beta| + \sum_{\gamma \neq \beta} \pi(\gamma)|\gamma| \geq$$
$$\geq |\beta| + \pi(\beta) - 1 + \sum_{\gamma \neq \beta} \pi(\gamma) = |\beta| - 1 + |\pi| = |\beta| + k - 1,$$

therefore $|\beta| \leq |\alpha| - k + 1$.

For the inverse inclusion, suppose β in D_α with $|\beta| \leq |\alpha| - k + 1$ and put $\gamma = \alpha - \beta$. Then γ has non-negative coordinates and $|\gamma| \geq k - 1$. By the Lemma 2.1 b), there is a partition τ of γ with $|\tau| = k - 1$. But then $\pi = \tau + \delta_\beta$ is a partition of α with $|\pi| = |\tau| + 1 = k$ and $\pi(\beta) \neq 0$, and the proof is complete. □

We can now conclude this section by giving the extended Bell polynomial form of the Faà di Bruno formula for partial derivatives. The sum over partitions on right side of (10) can be regrouped by cardinal and written as

$$\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\alpha} g(f(\mathbf{z})) = \sum_{k=1}^{|\alpha|} g^{(k)}(f(\mathbf{z})) \sum_{\pi \in P_{\alpha,k}} \frac{\alpha!}{\pi!} \prod_{\beta \in D_{\alpha,k}} \left(\frac{\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\beta} f(\mathbf{z})}{\beta!}\right)^{\pi(\beta)} =$$
$$= \sum_{k=1}^{|\alpha|} g^{(k)}(f(\mathbf{z})) B_{\alpha,k}\left(\left(\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\beta} f(\mathbf{z})\right)_{\beta \in D_{\alpha,k}}\right),$$

where the extended Bell polynomials $B_{\alpha,k}$ depend on a family $(x_\beta)_{\beta \in D_{\alpha,k}}$ of variables indexed over the domain $D_{\alpha,k}$ described in Proposition 1, and are defined by:

$$B_{\alpha,k}\left((x_\beta)_{\beta \in D_{\alpha,k}}\right) = \sum_{\pi \in P_{\alpha,k}} \frac{\alpha!}{\pi!} \prod_{\beta \in D_{\alpha,k}} \left(\frac{x_\beta}{\beta!}\right)^{\pi(\beta)}. \tag{14}$$

3. Vector-valued partitions of vectors and Faà di Bruno extensions

In this section we deal with compositions in the general case $f : \mathbb{R}^p \to \mathbb{R}^q$ and $g : \mathbb{R}^q \to \mathbb{R}$. As one could expect, the proper extended partitions in this case should be partitions of vectors in $\mathbb{N}^p$, as described in the previous section, but valued in $\mathbb{N}^q$, as in the framework of Adomian decompositions ([1], [3]). The natural definition is the following:

Definition 2. *Let $p \geq 1$, $q \geq 1$ be two integers and α a vector in $\mathbb{N}^p$. We will call a q-valued partition (or simply a q-partition) of the vector α any map*

$$\pi = (\pi_1, \ldots, \pi_q) : \mathbb{N}^p \setminus \{\theta\} \to \mathbb{N}^q$$

satisfying

$$\sum_{j=1}^{q}\sum_{\beta}\beta\pi_j(\beta)=\alpha.$$

The set of the q-partitions of α will be denoted P_α^q.

For any q-partition $\pi \in P_\alpha^q$, the components π_j are partitions of the vectors $\eta^j=\sum_\beta \beta\pi_j(\beta)$, with $\eta^1+\cdots+\eta^j=\alpha$. Thus the set P_α^q of the q-partitions of α can be regarded as a union of cartesian products of partition sets:

$$P_\alpha^q=\bigcup_{\substack{\eta^1,\ldots,\eta^q\in\mathbb{N}^p\\ \eta^1+\cdots+\eta^q=\alpha}} P_{\eta^1}\times P_{\eta^2}\times\cdots\times P_{\eta^q} \tag{15}$$

For $\pi \in P_\alpha^q$, the *cardinal* of π is defined as $|\pi|=\sum_{j=1}^q|\pi_j|=\sum_{j=1}^q\sum_\beta\pi_j(\beta)$. We also denote $\pi!=\prod_{j=1}^q\pi_j!$.

Let us consider now a vector-valued power series $f(\mathbf{z})=(f_1(\mathbf{z}),\ldots,f_q(\mathbf{z}))$ in p variables

$$f_j(\mathbf{z})=\sum_{\alpha\in\mathbb{N}^p}a_{j,\alpha}\mathbf{z}^\alpha \qquad (1\le j\le q)$$

and a scalar power series $g(\mathbf{w})=\sum_{\gamma\in\mathbb{N}^q}b_\gamma\mathbf{w}^\gamma$ in q variables.

We wish to compute the coefficients c_α of the composed power series

$$g(f(\mathbf{z}))=\sum_{\alpha\in\mathbb{N}^p}c_\alpha\mathbf{z}^\alpha.$$

For this, fix $\gamma=(\gamma_1,\ldots,\gamma_q)\in\mathbb{N}^q\setminus\{\theta\}$ and consider the composition of the single term $\beta_\gamma\mathbf{w}^\gamma$ of g with f, expanded as:

$$\underbrace{b_\gamma(\sum_\beta a_{1,\beta}\mathbf{z}^\beta)\cdots(\sum_\beta a_{1,\beta}\mathbf{z}^\beta)}_{\gamma_1}\cdots\underbrace{(\sum_\beta a_{q,\beta}\mathbf{z}^\beta)\cdots(\sum_\beta a_{q,\beta}\mathbf{z}^\beta)}_{\gamma_q}. \tag{16}$$

Fix now $\alpha\in\mathbb{N}^p$ and let us identify the terms in $\mathbf{z}^\alpha$ in the above expansion by the means of the q-partitions of α. Let us fix a q-partition π of α, such that $|\pi_j|\le\gamma_j$ for $1\le j\le q$. For every multi-degree $\beta\in\mathbb{N}^p$, each number $\pi_j(\beta)$ indicates how many times the term in z^β was chosen within the j-th group of parentheses in (16). Inside each group, the $\pi_j(\beta)$ parentheses can be chosen in $\gamma_j!/\Big(\gamma_j-|\pi_j|)!\prod_\beta\pi_j(\beta)!\Big)$ manners. By taking products over j, then summing up over all such q-partitions and all γ one obtains:

Theorem 3.1. *(extended Faà di Bruno formula for multiple power series) The coefficients of the composition $g\circ f$ are:*

$$c_\alpha=\sum_{\pi\in P_\alpha^q}\frac{\prod_{j=1}^q\prod_\beta a_{j,\beta}^{\pi_j(\beta)}}{\pi!}\left(\sum_{\substack{\gamma\in\mathbb{N}^q\\ \gamma_j\ge|\pi_j|}}b_\gamma\gamma!\prod_{j=1}^q\frac{a_{j,\theta}^{\gamma_j-|\pi_j|}}{(\gamma_j-|\pi_j|)!}\right) \tag{17}$$

As in the scalar case, the version for partial derivatives of the above formula can be obtained by replacing whole power series with partial Taylor series and a remainder, and setting $a_{1,\theta} = \cdots = a_{q,\theta} = 0$:

Theorem 3.2. *Let $\alpha \in \mathbb{N}^p \setminus \{\theta\}$, $f = (f_1, \ldots, f_q)$ a vector-valued function of p variables that admits partial derivatives up to the multi-order α at $\mathbf{z}_0$ and g a q-variable function admitting partial derivatives up to the order $|\alpha|$ at $f(\mathbf{z}_0)$. Then:*

$$\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\alpha} g(f(\mathbf{z}_0)) = \sum_{\pi \in P_{\alpha}^{q}} \frac{\alpha!}{\pi!} \frac{\partial^{|\pi|} g}{\partial w_1^{|\pi_1|} \ldots \partial w_q^{|\pi_q|}} (f(\mathbf{z}_0)) \prod_{j=1}^{q} \prod_{\beta} \left(\frac{\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\beta} f_j(\mathbf{z}_0)}{\beta!} \right)^{\pi_j(\beta)} \tag{18}$$

Before emphasizing the extended Bell polynomial form of (18), let us have a closer look at q-partitions with coordinatewise-specified cardinals. For $\alpha \in \mathbb{N}^p$ and $\kappa = (\kappa_1, \ldots, \kappa_q)$ in $\mathbb{N}^q$, we denote by $P_{\alpha,\kappa}^{q}$ the set of all the q-partitions π of α with $|\pi_j| = \kappa_j$, $1 \leq j \leq q$.

Proposition 2. *Let $p, q \geq 1$, $\alpha \in \mathbb{N}^p$ and $\kappa = (\kappa_1, \ldots, \kappa_q) \in \mathbb{N}^q$.*
a) The set $P_{\alpha,\kappa}^{q}$ is not empty if and only if $|\kappa| \leq |\alpha|$.
b) Suppose $|\kappa| \leq |\alpha|$ and let

$$V_{\alpha,\kappa} = \{(\eta^1, \ldots, \eta^q) \in (\mathbb{N}^p)^q : \sum_{j=1}^{q} \eta^j = \alpha, |\eta^j| \geq \kappa_j, 1 \leq j \leq q\}. \tag{19}$$

Then:

$$P_{\alpha,\kappa}^{q} = \bigcup_{(\eta^1, \ldots, \eta^q) \in V_{\alpha,\kappa}} P_{\eta^1,\kappa_1} \times P_{\eta^2,\kappa_2} \times \cdots \times P_{\eta^q,\kappa_q} \tag{20}$$

The assertion b) is straightforward from (15) and the Lemma 2.1 a), as the "only if" part of a). By (20) and the Lemma 2.1 b), the fact that $P_{\alpha,\kappa}^{q}$ is not empty when $|\alpha| \geq |\kappa|$ is equivalent to the fact that the set $V_{\alpha,\kappa}$ is not empty under the same hypothesis. This follows clearly from the following lemma which gives a "constructive" description of this latter set.

Lemma 3.1. *Let $\alpha \in \mathbb{N}^p$ and $\kappa = (\kappa_1, \ldots, \kappa_q) \in \mathbb{N}^q$ with $|\kappa| \leq |\alpha|$. Any q-tuple of vectors $(\eta^1, \ldots, \eta^q)$ in $V_{\alpha,\kappa}$ satisfies the following conditions:*

$(C_1) : \eta^1 \in D_{\alpha}, \quad \kappa_1 \leq |\eta^1| \leq |\alpha| - \sum_{l=2}^{q} \kappa_l,$

$(C_j) : \eta^j \in D_{\alpha - \sum_{l=1}^{j-1} \eta^l}, \quad \kappa_j \leq |\eta^j| \leq |\alpha| - \sum_{l=2}^{j-1} |\eta^l| - \sum_{l=j+1}^{q} \kappa_l, \quad (2 \leq j \leq q-1)$

$(C_q) : \eta^q = \alpha - \eta^1 - \cdots - \eta^{q-1}.$

Conversely, if $|\kappa| \leq |\alpha|$, then there is at least one vector η^1 satisfying (C_1). For any such η^1, there is at least one vector η^2 verifying (C_2) and so on. Any q-tuple $(\eta^1, \ldots, \eta^q)$ chosen this way lies in $V_{\alpha,\kappa}$.

Proof. For the direct implication, let $(\eta^1, \ldots, \eta^q)$ in $V_{\alpha,\kappa}$, so $\eta^1 + \cdots + \eta^q = \alpha$ and $|\eta^j| \geq \kappa_j$. This immediately implies (C_q), the hyper-rectangle conditions and the left inequalities in $(C_1), \ldots, (C_{q-1})$. The right inequalities follow from

$$|\eta^j| = |\alpha| - \sum_{l<j} |\eta^l| - \sum_{l>j} |\eta^l| \leq |\alpha| - \sum_{l<j} |\eta^l| - \sum_{l>j} \kappa_l \qquad (2 \leq j \leq q-1).$$

Conversely, the double inequality $|\kappa_1| \leq |\alpha| - \sum_{l=2}^{q} \kappa_l \leq |\alpha|$ shows that there is at least one vector η^1 satisfying (C_1). Fix $j \leq q-1$ and suppose we chose by recurrence $\eta^1, \ldots, \eta^{j-1}$ such that η^l satisfies (C_l) for $1 \leq l \leq j-1$. Then from the right inequality in (C_{j-1})

$$|\eta^{j-1}| \leq \sum_{l=2}^{j-2} |\eta^l| - \sum_{l=j}^{q} \kappa_l = \sum_{l=2}^{j-2} |\eta^l| - \kappa^j - \sum_{l=j+1}^{q} \kappa_l$$

it follows that

$$\kappa_j \leq |\alpha| - \sum_{l=2}^{j-1} |\eta^l| - \sum_{l=j+1}^{q} \kappa_l \leq |\alpha| - \sum_{l=2}^{j-1} |\eta^l|,$$

which means there is at least one vector η_j satisfying (C_j).

Finally, suppose that $\eta^1, \ldots, \eta^{q-1}$ were chosen. Since $\eta^{q-1} \in D_{\alpha - \sum_{l=1}^{q-1} \eta^l}$, the vector $\eta^q = \alpha - \sum_{l=1}^{q-1} \eta^l$ has non-negative coordinates. In addition, the right inequality in (C_{q-1}) implies $\kappa_q \leq |\alpha| - \sum_{l=1}^{q-1} |\eta^l| = |\eta^q|$, which, together with the left inequalities in $(C_1), \ldots, (C_{q-1})$, show that $(\eta^1, \ldots, \eta^q)$ lies in $V_{\alpha,\kappa}$, and the proof is complete. $\square$

As in the previous section, the next proposition regards the index domain for the variables of the extended Bell polynomials:

Proposition 3. *Let $\alpha \in \mathbb{N}^p \setminus \{\theta\}$ and $\kappa = (\kappa_1, \ldots, \kappa_q) \in \mathbb{N}^q \setminus \{\theta\}$ with $|\kappa| \leq |\alpha|$. Then*

$$\bigcup_{\pi \in P^q_{\alpha,\kappa}} supp(\pi) = D_{\alpha,|\kappa|} = \{\beta \in D_\alpha : |\beta| \leq |\alpha| - |\kappa| + 1\}. \qquad (21)$$

Proof. For the direct inclusion, let $\beta \in \mathbb{N}^p$ such that $\pi_j(\beta) \neq 0$ for some $\pi \in P^q_{\alpha,\kappa}$ and some $j \in \{1, \ldots, q\}$. Then

$$\begin{aligned} |\alpha| &= \sum_{l=1}^{q} \sum_{\gamma} |\gamma| \pi_l(\gamma) = \sum_{l=1}^{q} \sum_{\gamma \neq \beta} |\gamma| \pi_j(\gamma) + \sum_{l \neq j} |\beta| \pi_l(\beta) + |\beta| \pi_j(\beta) \geq \\ &\geq |\beta| + (\pi_j(\beta) - 1) + \sum_{l=1}^{q} \sum_{\gamma \neq \beta} \pi_j(\gamma) + \sum_{l \neq j} \pi_l(\beta) = |\beta| + |\pi| - 1, \end{aligned}$$

thus $|\beta| \leq |\alpha| - |\pi| + 1 = \beta \leq |\alpha| - |\kappa| + 1$, meaning that β lies in $D_{\alpha,|\kappa|}$.

Conversely, suppose $\beta \in D_\alpha$ with $|\beta| \leq |\alpha| - |\kappa| + 1$ and let $\gamma = \alpha - \beta$. Since $\kappa \neq \theta$, κ has at least one component, say $\kappa_j \geq 1$. Let then $\tilde{\kappa} = \kappa - \delta_{e_j}$, which satisfies $|\tilde{\kappa}| = |\kappa| - 1 \leq |\gamma|$. By the Proposition 2 b), there is at least one q-partition $\tilde{\pi}$ of γ with $|\tilde{\pi}_l| = \tilde{\kappa}_l$ for $1 \leq l \leq q$. Define then $\pi = (\pi_1, \ldots, \pi_q)$ by $\pi_l = \tilde{\pi}_l$ for $l \neq j$ and $\pi_j = \delta_\beta + \tilde{\pi}_j$. Then clearly π is a q-partition of α, $|\pi_l| = \kappa_l$ for $1 \leq l \leq q$ and in addition $\pi_j(\beta) \neq 0$, which completes the proof. $\Box$

We can now reformulate the Faà di Bruno extension in the Theorem 3.2 in terms of extended Bell polynomials:

Theorem 3.3. *Under the hypotheses of the Theorem 3.2,*

$$\left(\frac{\partial}{\partial \mathbf{z}}\right)^\alpha g \circ f = \sum_{\substack{\kappa \in \mathbb{N}^q \setminus \{\theta\} \\ |\kappa| \leq |\alpha|}} \left[\left(\frac{\partial}{\partial \mathbf{w}}\right)^\kappa g\right](f) B_{\alpha,\kappa}\left(\left(\left(\frac{\partial}{\partial \mathbf{z}}\right)^\beta f_j\right)_{\substack{\beta \in D_{\alpha,|\kappa|} \\ 1 \leq j \leq q}}\right),$$

where the extended Bell polynomials $B_{\alpha,\kappa}$ depend of a family of variables indexed over the set $\{1, \ldots, q\} \times D_{\alpha,|\kappa|}$ and are defined by:

$$B_{\alpha,\kappa}\left((x_{j,\beta})_{\substack{\beta \in D_{\alpha,|\kappa|} \\ 1 \leq j \leq q}}\right) = \sum_{\pi \in P^q_{\alpha,\kappa}} \frac{\alpha!}{\pi!} \prod_{j=1}^{q} \prod_{\beta \in D_{\alpha,|\kappa|}} \left(\frac{x_{j,\beta}}{\beta!}\right)^{\pi_j(\beta)}.$$

We end this section by pointing out how the algorithmic construction of the classes of q-valued vector partitions P^q_α and $P^q_{\alpha,\kappa}$ can be reduced to the construction of the classes P_α and $P_{\alpha,k}$ of single-valued vector partitions.

It is clear from by (15) that constructing P^q_α means constructing each P_β with $\beta \in D_\alpha$ and generating the set $\{\eta_1, \ldots, \eta_q \in \mathbb{N}^p : \eta^1 + \cdots + \eta^q = \alpha\}$. This latter set can be easily generated by a standard back-tracking scheme: select any vector $\eta^1 \in D_\alpha$, for each such choice select any vector $\eta^2 \in D_{\alpha-\eta^1}$ and so on up to the choice of $\eta^{q-1} \in D_{\alpha-\eta^1-\cdots-\eta^{q-2}}$.

Concerning the set $P^q_{\alpha,\kappa}$, by the Proposition 2 its construction means generating the sets P_{β,κ_j} for $\beta \in D_\alpha$, and generating the set $V_{\alpha,\kappa}$ defined in (19), whose algorithm is clearly described by the Lemma 3.1.

The most delicate part seems to be the construction of the classes P_α themselves, which will be the topic of the next section.

4. An algorithm for generating vector partitions

This section provides an efficient recursive algorithm to generate the sets of vector partitions P_α.

The generic idea of a recursion scheme in this framework is to give a set of rules for transforming the partitions of some vector $\alpha \in \mathbb{N}^p$ - which are supposed to have already been generated -into partitions of a vector $\beta \in \mathbb{N}^p$ which is "greater" than α with respect to some (partial) order on $\mathbb{N}^p$, such as the natural partial order defined by $\alpha \leq \beta$ iff $\alpha \in D_\beta$. For the recursion to work, it obviously suffices to define rules to transform for instance the partitions of any vector α into partitions of the vectors $\alpha + e_1, \dots \alpha + e_p$, where, as in the previous sections $(e_i)_{1\le i\le p}$ stands for the canonical base in $\mathbb{R}^p$.

In order to formalize such a set of rules, we will call a *transition* from P_α to P_β any set-valued map $T_{\alpha\to\beta} : P_\alpha \to \mathcal{P}(P_\beta)$, associating to each partition of α a set of partitions of β. For such a map and for $\pi \in P_\alpha$, we will say that the partitions of β in $T_{\alpha\to\beta}(\pi)$ are *generated by* π.

For a transition-based algorithm to work, the minimal requirement is that every partition of β should be generated by some partition of α, in other words $\cup_{\pi\in P_\alpha} T_{\alpha\to\beta}(\pi) = P_\beta$. If this happens, we say that the transition $T_{\alpha\to\beta}$ is "onto".

An natural efficiency requirement for a transition-based algorithm is that one same partition of β should not be generated several times, i.e. from several partitions of α. Formally this means that $T_{\alpha\to\beta}(\pi_1) \cap T_{\alpha\to\beta}(\pi_2) = \emptyset$ whenever $\pi_1 \neq \pi_2$. If this happens we will say that $T_{\alpha\to\beta}$ is *non-redundant.*

Therefore an efficient algorithm, in the sense defined above, means a family of non-redundant onto transitions $(T_{\alpha\to\alpha+e_i})_{\alpha\in\mathbb{N}^p, 1\le i\le p}$.

In the one dimensional case, a simple and well-known recursive algorithm to generate the partitions of $n+1$ from the partitions of n is the following: suppose π is an arbitrary partition of n,

$$n = \underbrace{1 + \cdots + 1}_{\pi(1)} + j_1 + \cdots ,$$

where j_1 is the smallest term in the partition greater than 1 (if any). This partition always generates at least the partition

$$n + 1 = \underbrace{1 + \cdots + 1}_{\pi(1)+1} + j_1 + \cdots$$

of $n+1$, and furthermore, either if j_1 is missing (i.e. $\pi(1) = n$), or if $\pi(1)+1 \le j_1$, it also generates the additional partition

$$n + 1 = (\pi(1) + 1) + j_1 + \dots .$$

The transitions associated to this algorithm are:

$$T_{n\to n+1}(\pi) = \begin{cases} \{\pi + \delta_1\}, & 1 + \pi(1) > m_\pi \\ \{\pi + \delta_1,\ \pi - \pi(1)\delta_1 + \delta_{1+\pi(1)}\}, & 1 + \pi(1) \le m_\pi \end{cases} , \qquad (n \in \mathbb{N}^*) \tag{22}$$

where m_π is the minimum of $supp(\pi) \setminus \{1\}$ if this set is not empty, or $m_\pi = n+1$ otherwise (as in the previous sections, δ denotes Dirac functions).

In the following we give a p-dimensional extension of this algorithm by constructing a transition family $(T_{\alpha\to\alpha+e_i})_{\alpha\in\mathbb{N}^p\setminus\{\theta\}, 1\le i\le p}$ that extends (22).

Fix $\alpha \in \mathbb{N}^p \setminus \{\theta\}$, $1 \le i \le d$ and partition π in P_α, and let us describe the members of $T_{\alpha\to\alpha+e_i}(\pi)$. Similarly to (22), $T_{\alpha\to\alpha+e_i}(\pi)$ will at least contain the partition $\pi + \delta_{e_i}$.

To specify the possible other members of $T_{\alpha\to\alpha+e_i}(\pi)$ we need to endow $\mathbb{N}^p$ with a suitable lexicographic order. We recall that any permutation σ of $\{1,...,p\}$ gives rise to a total order denoted $\overset{\sigma}{<}$ on $\mathbb{N}^p$, by

$$\alpha \overset{\sigma}{<} \beta \Leftrightarrow (\exists m \leq p)(\forall k < m)(\alpha_{\sigma(k)} = \beta_{\sigma(k)}) \wedge (\alpha_{\sigma(m)} < \beta_{\sigma(m)}), \tag{23}$$

representing a lexicographic order for which the coordinate $\sigma(1)$ has the highest priority, then the coordinate $\sigma(2)$ etc. In our case we need any lexicographic order $\overset{\sigma}{<}$ with $\sigma(p) = i$, i.e. for which the i^{th} coordinate has the least priority. This makes in fact e_i the smallest vector in $\mathbb{N}^p \setminus \{\theta\}$. We will fix any of the $(p-1)!$ permutations σ with this property, and simply write $<$ instead of $\overset{\sigma}{<}$.

With respect to such a fixed order, put

$$S_\pi = \{\beta = (\beta_1 \ldots, \beta_p) \in supp(\pi) : \beta > e_i, \beta_i \neq 0\} \tag{24}$$

and

$$m_\pi = \begin{cases} \min\, S_\pi, & S_\pi \neq \emptyset \\ \alpha + e_i, & S_\pi = \emptyset, \end{cases} \tag{25}$$

The minimal vector m_π plays a similar role as in the one-dimensional case (22). In contrast, here there might be more than one additionally generated partitions, depending on the (possibly empty) set:

$$K_\pi = \{\beta \in supp(\pi) \cup \{\theta\} : \beta_i = 0, e_i < \beta + (1 + \pi(e_i))e_i \leq m_\pi\} \tag{26}$$

Each vector β in K_π gives rise to a newly generated partition π_β of $\alpha + e_i$ defined by:

$$\pi_\beta = \pi - \pi(e_i)\delta_{e_i} - \delta_\beta + \delta_{\beta+(1+\pi(e_i))e_i} \tag{27}$$

The formula above means that we remove from the partition π the $\pi(e_i)$ occurrences of the vector e_i, together with the vector β, and replace them by the vector $\beta + (1 + \pi(e_i))e_i$.

Summing up, our algorithm is based on the transitions $T_{\alpha\to\alpha+e_i}$ defined by:

$$T_{\alpha\to\alpha+e_i}(\pi) = \{\delta_{e_i} + \pi\} \cup \{\pi_\beta : \beta \in K_\pi\} \tag{28}$$

Note that $\delta_{e_i} + \pi$ is the only partition generated by π that has the vector e_i in its support, while for the partitions of the type π_β we have $\pi_\beta(e_i) = 0$.

Here is an example how the algorithm works in the case $p = 2$. In this case the choice of the lexicographic order is unique. The table below illustrates how the partitions of $(2, 3)$ are obtained by (28) from the partitions of $(1, 3)$. In the example, instead of an additive notation we rather use a multiplicative notation (more concise and easier to read), i.e. the vectors in the support of each partition are simply appended and their multiplicities are indicated as exponents.

$\pi \in P_{(1,3)}$	S_π	m_π	K_π	$T_{(1,3)\to(2,3)}(\pi)$
$(1,0)(0,1)^3$	$\emptyset$	$(2,3)$	$\{(0,0),$ $(0,1)\}$	$(1,0)^2(0,1)^3$ $(2,0)(0,1)^3$ $(2,1)(0,1)^2$
$(1,1)(0,1)^2$	$\{(1,1)\}$	$(1,1)$	$\{(0,1)\}$	$(1,0)(1,1)(0,1)^2$ $(1,1)^2(0,1)$
$(1,0)(0,3)$	$\emptyset$	$(2,3)$	$\{(0,0),$ $(0,3)\}$	$(1,0)^2(0,3)$ $(2,0)(0,3)$ $(2,3)$
$(1,3)$	$\{(1,3)\}$	$(1,3)$	$\emptyset$	$(1,0)(1,3)$ -
$(1,0)(0,1)(0,2)$	$\emptyset$	$(2,3)$	$\{(0,0),$ $(0,1),$ $(0,2)\}$	$(1,0)^2(0,1)(0,2)$ $(2,0)(0,1)(0,2)$ $(2,1)(0,2)$ $(2,2)(0,1)$
$(1,2)(0,1)$	$\{(1,2)\}$	$(1,2)$	$\{(0,1)\}$	$(1,0)(1,2)(0,1)$ $(1,1)(1,2)$
$(1,1)(0,2)$	$\{(1,1)\}$	$(1,1)$	$\emptyset$	$(1,0)(1,1)(0,2)$ -

Let us prove now that the transitions $T_{\alpha\to\alpha+e_i}$ defined by (28) are non-redundant and "onto" for any $\alpha \in \mathbb{N}^p \setminus \{\theta\}$ and $1 \le i \le p$. The next two lemmas contain the main arguments for the non-redundancy and "onto" parts respectively.

Lemma 4.1. *Let $\alpha \in \mathbb{N}^p \setminus \{\theta\}$, $1 \le i \le p$, $\pi \in P_\alpha$ and $\beta \in K_\pi$. If $\tau = \pi_\beta$ then*

$$m_\tau = \beta + (1 + \pi(e_i))e_i. \tag{29}$$

Proof. Put $\gamma = \beta + (1 + \pi(e_i))e_i$. We show that γ belongs to S_τ. Clearly $\gamma_i = 1 + \pi(e_i) > 0$. Since $\beta \in K_\pi$ it follows by (26) that $\gamma > e_i$. Also,

$$\begin{aligned}\tau(\gamma) &= \pi_\beta(\gamma) = \pi(\gamma) - \pi(e_i)\delta_{e_i}(\gamma) - \delta_\beta(\gamma) + \delta_\gamma(\gamma) = \\ &= \pi(\gamma) + 1 > 0\end{aligned}$$

so $\gamma \in supp(\tau)$. Therefore, by (24), γ lies in S_τ. This means in particular that S_τ is not empty, so $m_\tau = min\, S_\tau \le \gamma$. But on the other hand $\gamma \le m_\tau$ because β is in K_π. In conclusion $m_\tau = \beta + (1 + \pi(e_i))e_i$ and the proof is complete.□

Lemma 4.2. *Let $\alpha \in \mathbb{N}^p \setminus \{\theta\}$, $1 \le i \le p$ and $\tau \in P_{\alpha+e_i}$ such that $\tau(e_i) = 0$. Let m_τ as in (25), let k be the i-th component of m_τ and $\omega = m_\tau - ke_i$. Put*

$$\pi = \tau - \delta_{m_\tau} + \delta_\omega + (k-1)\delta_{e_i}. \tag{30}$$

Then:
a) ω lies in K_π;
b) $\tau = \pi_\omega$.

Proof. Let us first show that S_τ is not empty. If we suppose that $S_\tau = \emptyset$, this implies that any vector $\beta \in supp(\tau)$ such that $\beta > e_i$ necessarily has $\beta_i = 0$. Since $e_i \notin supp(\tau)$, any $\beta \in supp(\tau)$, satisfies $\beta_i = 0$. But since τ is a partition in $P_{\alpha+e_i}$, we have $\alpha + e_i = \sum_{\beta \in supp(\tau)} \tau(\beta)\beta$, which cannot hold because the right side has the i^{th} coordinate equal to zero, while the i^{th} coordinate of the left side has at least 1. This shows that S_τ is not empty.

It follows then from (25) that $m_\tau = min\, S_\tau$. Write $m_\tau = ke_i + \omega$, with $\omega_i = 0$. Since m_τ lies in S_τ we have $k \geq 1$ and $m_\tau > e_i$. Thus $k \geq 1$ if $\omega \neq 0$ and $k \geq 2$ if $\omega = 0$.

The next step is to prove that $S_\pi \subseteq S_\tau$. Indeed, if $\beta \in S_\pi$ then (24) implies $\pi(\beta) \neq 0, \beta > e_i$ and $\beta_i \neq 0$, so we only have to check that $\tau(\beta) \neq 0$. This is true because

$$\begin{aligned} \tau(\beta) &= \pi(\beta) + \delta_{m_\tau}(\beta) - \delta_\omega(\beta) - (k-1)\delta_{e_i}(\beta) = \\ &= \pi(\beta) + \delta_{m_\tau}(\beta) \geq \pi(\beta) > 0. \end{aligned}$$

The fact that $S_\pi \subseteq S_\tau$ implies $m_\tau = min\{S_\tau\} \leq m_\pi$ (see (25) and (28)).

Observe also that

$$\pi(e_i) = \tau(e_i) - \delta_{m_\tau}(e_i) + \delta_\omega(e_i) + (k-1)\delta_{e_i}(e_i) = \tau(e_i) + k - 1 \qquad (31)$$

We show now that ω lies in K_π. Clearly $\omega_i = 0$, and from (31) we obtain that

$$\omega + (1 + \pi(e_i))e_i \quad = \quad \omega + (1 + \tau(e_i) + k - 1)e_i = \omega + ke_i = m_\tau,$$

and so $e_i < \omega + ke_i = m_\tau \leq m_\pi$.

To prove a) it is now enough to see that $\omega \in supp(\pi) \cup \{0\}$. Suppose $\omega \neq 0$. Then

$$\pi(\omega) = \tau(\omega) - \delta_{m_\tau}(\omega) + \delta_\omega(\omega) + (k-1)\delta_{e_i}(\omega) = \tau(\omega) + 1 > 0.$$

Therefore $\omega \in K_\pi$, so a) is proved.

Now b) follows from (27) and (31):

$$\begin{aligned} \pi_\omega &= \pi - \pi(e_i)\delta_{e_i} - \delta_\omega + \delta_{\omega+(1+\pi(e_i))e_i} = \\ &= \tau - \delta_{m_\tau} + \delta_\omega + (k-1)\delta_{e_i} - (\tau(e_i) + k - 1)\delta_{e_i} - \delta_\omega + \delta_{m_\tau} = \\ &= \tau. \square \end{aligned}$$

We can now prove the main result:

Theorem 4.1. *For any $\alpha \in \mathbb{N}^p \setminus \{\theta\}$ and $1 \leq i \leq p$, $T_{\alpha \to \alpha+e_i}$ defined in (28) is a non-redundant transition from P_α "onto" $P_{\alpha+e_i}$.*

Proof. Let us show first that $T_{\alpha\to\alpha+e_i}$ is "onto". Let τ be a partition in $P_{\alpha+e_i}$.

If $\tau(e_i) \neq 0$ then τ belongs to $T_{\alpha\to\alpha+e_i}(\pi)$, where $\pi = \tau - \delta_{e_i}$.

If $\tau(e_i) = 0$, it follows from Lemma 4.2 that $\tau = \pi_u$ for some $\pi \in P_\alpha$ and $u \in K_\pi$. By (28) this implies that $\tau \in T_{\alpha\to\alpha+e_i}(\pi)$. Therefore $T_{\alpha\to\alpha+e_i}$is "onto".

We show now that $T_{\alpha\to\alpha+e_i}$ is non-redundant. Let π and π' two partitions in P_α and suppose $\tau \in T_{\alpha\to\alpha+e_i}(\pi) \cap T_{\alpha\to\alpha+e_i}(\pi')$.

If $\tau(e_i) \neq 0$ then necessarily $\tau = \pi + \delta_{e_i} = \pi' + \delta_{e_i}$, therefore $\pi = \pi'$.

Suppose now that $\tau(e_i) = 0$, thus $\tau = \pi_\beta = \pi'_{\beta'}$ for some $\beta \in K_\pi$ and $\beta' \in K_{\pi'}$. From Lemma 4.1 it follows that $m_\tau = \beta + (1+\pi(e_i))e_i = \beta' + (1+\pi'(e_i))e_i$ which implies that $\beta = \beta'$ and $\pi(e_i) = \pi'(e_i)$. Then

$$\begin{aligned} 0 &= \pi_\beta - \pi'_{\beta'} = \pi - \pi(e_i)\delta_{e_i} - \delta_\beta + \delta_{\beta+(1+\pi(e_i))e_i} - \\ &\quad - \left(\pi' - \pi'(e_i)\delta_{e_i} - \delta_{\beta'} + \delta_{\beta'+(1+\pi'(e_i))e_i}\right) = \\ &= \pi - \pi'. \end{aligned}$$

Therefore $T_{\alpha\to\alpha+e_i}$ is non-redundant and the proof is complete.□

In practical implementations, normally the partitions should be generated only once (for as many vectors as necessary) and then stored. While recursively creating these lists, new partitions can be arranged, as they are generated, in the increasing order of their cardinals, in order to obtain the list P_α as the concatenation of the lists $P_{\alpha,1}$,$P_{\alpha,2}$ etc. In this way one can easily form if necessary any specific extended Bell polynomial (14).

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