The Ulam-Hammesley problem for permutations and partial orders

(~~Heapability of partial orders~~)

Gabriel Istrate
West University of Timişoara, Romania
gabrielistrate@acm.org
http://tcs.ieat.ro

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Disclaimers

- This presentation: Combines results from multiple papers (CPM'2015, DCFS'2016), plus manuscript in progress.
- manuscript: János Balogh (Szeged), Cosmin Bonchiş , Diana Diniş, G.I. (Timişoara), and Ioan Todincã (Orleans).
- Results and ideas, not proofs !
- ... tried hard to make things simpler than they actually are.

Outline

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- The U-H. problem for partial orders
- Open problems.

Starting Point: *Longest Increasing Subsequence*

3 2 5 7 1 6 9

Patience sorting.

Another (greedy, also first-year) algorithm:

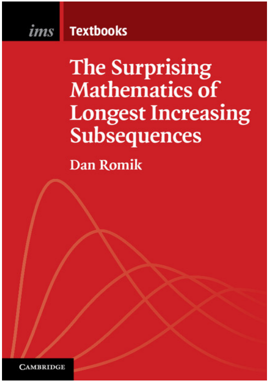
> Start (greedily) building decreasing piles. When not possible, start new pile.

Size of LIS = # of piles in patience sorting.

The Ulam-Hammersley problem (for random permutations)

What is the LIS of a random permutation ?

$$E_{\pi \in S_n}[LIS(\pi)] = 2\sqrt{n} \cdot (1 + o(1)).$$



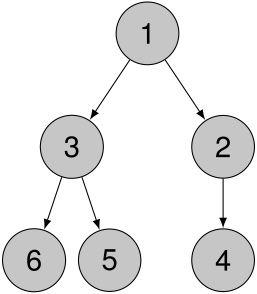
- Logan-Shepp (1977), Veršik-Kerov (1977), Aldous-Diaconis (1995)
- Very rich problem. Connections with nonequilibrium statistical physics and theory of Young tableaux

From (increasing) sequences to heaps

Byers, Heeringa, Mitzenmacher, Zervas (ANALCO'2011)

Sequence of integers A is heapable if it can be inserted into binary heap-ordered tree (not necessarily complete), always as leaf nodes.

Example: 1 3 2 6 5 4   Counterexample: 5 1 . . .



The Ulam-Hammersley problem for heapable sequences

- Simplest version trivial: $LHS(\pi) = n - o(n)$ (Byers et al.)
- (Dilworth, patience sorting): $LIS(\pi) =$ minimum number of decreasing sequences in a partition of π.

$HEAPS_k =$ minimum number of k-heapable sequences in a partition of π into such seqs.

Ulam-Hammersley problem for heapable sequences:

What is the scaling of $E_{\pi \in S_n}[HEAPS_k(\pi)], k \geq 2$?

A beautiful conjecture

For $k \geq 2$ there exists $\lambda_k > 0$ such that

$$\lim_{n \to \infty} \frac{E[HEAPS_k(\pi)]}{ln(n)} = \lambda_k$$

Moreover

$$\lambda_2 = \frac{1 + \sqrt{5}}{2}$$

is the golden ratio.

Status of the conjecture

- Some partial results.
- "Physics-like" nonrigorous argument, includes prediction for value of constant λ_k.
- Computations corroborated by experiments, "experimental mathematics" paper in progress.
- Follow-up work: Basdevant et al. (2016, 2017) rigorously establishes logarithmic scaling, but not the value of the constant.

Sample rigorous result: "Patience heaping"

> Theorem: The "Patience heaping" algorithm correctly computes the value of parameter $Heaps_k(\pi)$.

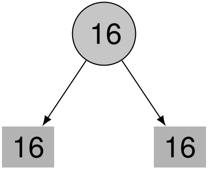
Proof idea:

By induction: The multiset of slots created by patience heaping is smaller ("dominates") the multiset of slots created by any other algorithm.

Consequently, when PatienceHeaping creates a new heap, so does any other algorithm.

Patience heaping

16, 25, 18, 2, 4, 35, 3, 7, 32, 9, 20



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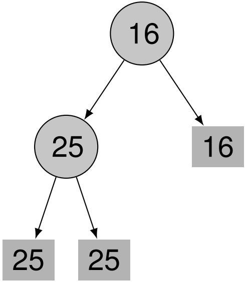

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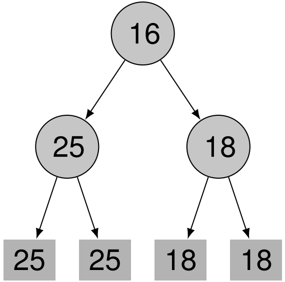

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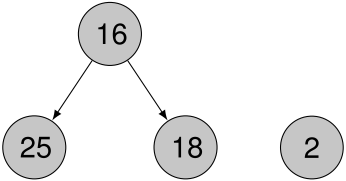

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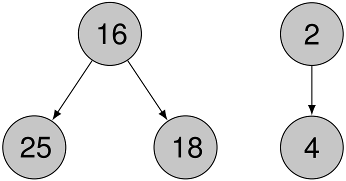

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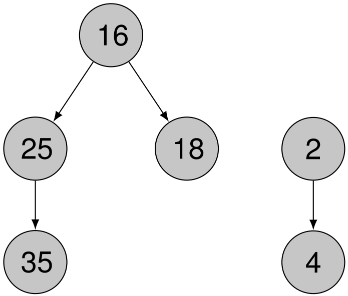

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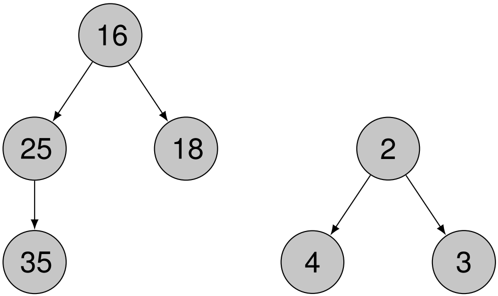

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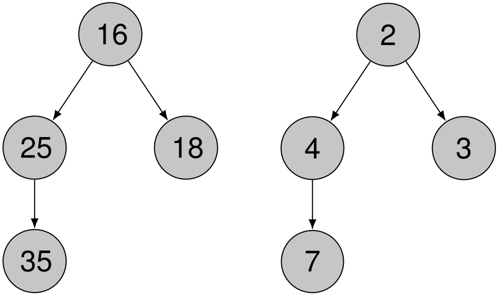

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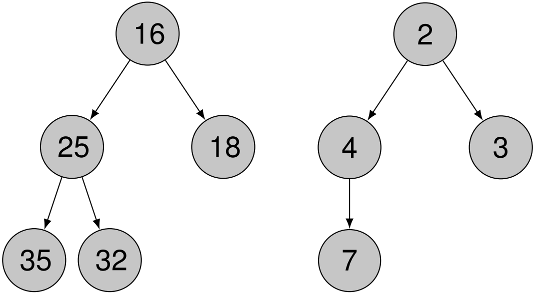

First result: Patience heaping

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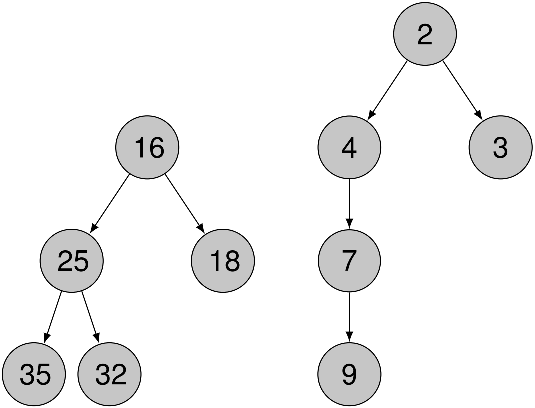

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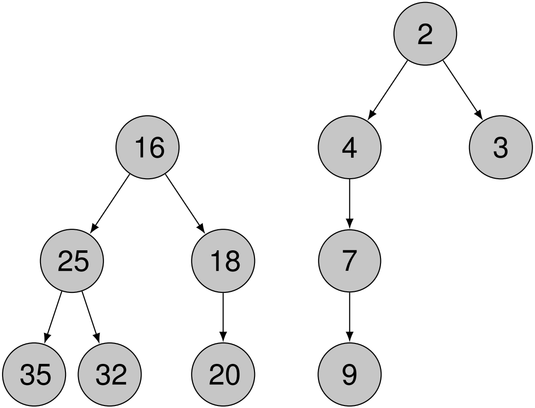


LIS and Hammersley's process

Top of piles in patience sorting = live particles in *Hammersley's process:*

- Particles: random real numbers $X_i \in (0, 1)$.
- Particle X_i kills closest live particle $X_j > X_i$ (if any)

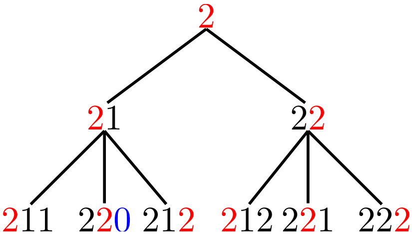
- studied in the area of interacting particle systems
- relative of a more famous process, the so-called Totally Asymmetric Exclusion Process (TASEP)

Aldous-Diaconis: Most illuminating proof of $E[LIS(\pi)] \sim 2\sqrt{n}$, analysis of the so-called hydrodynamic limit of Hammersley's process.

"Physics of patience heaping"

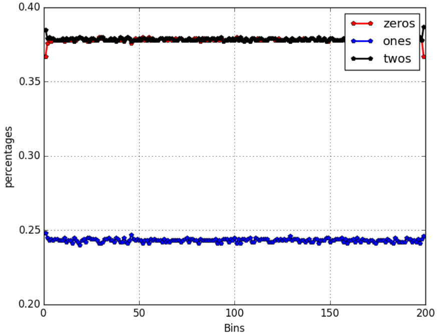
Hammersley's process with k lifelines (HAM_k):

- "Particles": random numbers $X_i \in (0, 1)$.
- each initially endowed with k lives.
- X_i removes one lifeline from closest live $X_j > X_i$ (if any)
- Combinatorially, $k = 2$: Words over alphabet $0, 1, 2$.
- Choose a random position. Put there a 2. Remove 1 from the closest nonzero digit to the right (if any).
- $E[\Delta(\#\text{ of heaps})] = 1 + E[\#\text{ of trailing zeros of } w]$



A "physicist's explanation" for the dynamics of HAM$_k$

- *$n \to \infty$: Limit of W_n* = compound Poisson process. $W_n =$ random string of 0,1,2 (densities c_0, c_1, c_2).
- *Assuming well mixing of digits* evolution equations $\to$ prediction on values of c_0, c_1, c_2.



- $c_0 = c_2 \sim \frac{3-\sqrt{5}}{2} \sim 0.381\ldots$, $c_1 \sim \sqrt{5} - 2 \sim 0.236\ldots$
- Distribution of trailing zeros: asymptotically geometric
- From this: $E[\Delta(\#\ heaps.)\ at\ stage\ n] \sim \frac{1+\sqrt{5}}{2} \cdot \frac{1}{(n+1)}$.

The plot thickens: intervals/posets

- Longest increasing subsequence of intervals: another first-year problem.
- Justicz, Scheinerman, Winkler (AMM 1990): random intervals on [0,1].

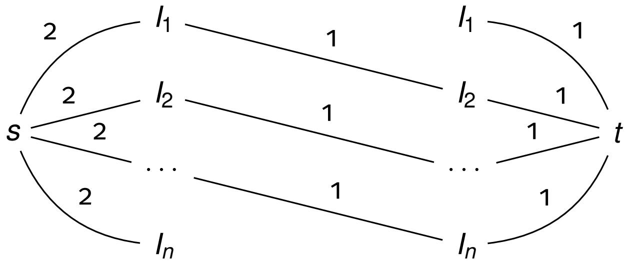
$$E[LIS(\pi)] \sim \frac{2\sqrt{n}}{\sqrt{\pi}}.$$

Theorem: PatienceHeaping still computes the decomposition of a sequence of intervals into a minimal number of heaps.

- Slot value of an interval: larger end.

Alternate solution

János Balogh (Szeged):



Theorem: number of heaps = n-maxflow.

Heapability and Dilworth's theorem.

- Width of poset: the minimum number of classes of a partition into chains.

Dilworth's Theorem: width= cardinality of largest antichain.

- k-chains: every item at most k direct successors.
- k-width(P): Minimum # classes of a partition into k-chains.

Theorem: The Network flow algorithm computes the k-width of arbitrary partial orders.

Essentially a generalization of (part of) the (network flow) proof of Dilworth's theorem.

IP computation of the k-width

Theorem

The following IP for k-width(P) has a totally unimodular matrix:

$$\begin{cases} max(1 + \sum_{p \prec q} X_{p,q}) \\ \sum_{q: p \prec q} X_{p,q} \leq k, \forall p \in P \\ \sum_{p: p \prec q} X_{p,q} \leq 1, \forall q \in P \\ X_{p,q} \in \{0, 1\} \end{cases}$$

Proof: if 1 instead of k on the right-hand side, IP for maximum matching in bipartite graph.

Further results on partial orders

THM: The k-width of a set of intervals is equal to that of the sequence of intervals obtained by listing intervals sorted by their right endpoint.

Trapezoid posets: induced by partial order among boxes in $\mathbf{R}^n$. Corresponding graphs: poly-time coloring, ind. set, etc.

Theorem: Greedy algorithm computes the k-width of trapezoid partial orders.

Theorem: For interval partial orders the longest heapable subsequence computable in poly-time.

Conclusions

Rich problem with many open questions:

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$$\lim_{n\to\infty} \frac{E[k\text{-width}(P)]}{n} = \frac{1}{k+1}.$$

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$$\lim_{n\to\infty} \frac{E[k\text{-width}(P)]}{n} = \frac{1}{k+1}.$$

- Heapability of random d-dimensional posets (DCFS'2016) (random model: Winkler, Bollobas and Winkler)

$$E[k\text{-width}(P)] = \Theta(\log^{d-1}(n)).$$

Thank you. Questions ?