Interactive Particle Systems, Explosive Random Walks on Hypergraphs and the WalkSAT algorithm

Gabriel Istrate[1], **Mircea Marin**, **Cosmin Bonchiş**

West University of Timişoara and the eAustria Research Institute,
Bd. V. Pârvan 4, cam 045B, Timişoara, RO-300223, Romania.
email: {gistrate,cbonchis,mmarin}@info.uvt.ro

ABSTRACT

We study the problem of analyzing the expected running time of WalkSAT, a local search procedure for satisfiability problem, on instances of the k-XOR SAT problem. We obtain upper bounds on this expected running time by reducing the problem to settings amenable to classical techniques from drift analysis. A crucial ingredient of this reduction is the definition of hypergraph versions of interacting particle systems and random walks, notably generalizations of coalescing and annihilating random walks. The use of such models allows to show that the nature of the expected running time of WalkSAT depends on a structural parameter (we call *odd Cheeger time*) of the dual of the formula hypergraph.
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1. INTRODUCTION

Interacting particle systems are discrete dynamical systems, usually defined on lattices, studied intensely in Mathematical Physics [Lig04]. They can be investigated on finite graphs as well [DW83],[DW84], [Ald13] as finite Markov chains, and correspond via *duality* to certain types of random walks [AF14]. The analysis of these models can sometimes be used to bound the mixing time of certain Markov chains, e.g. (hyper)graph coloring procedures [DW84, CT13].

[1]corresponding author

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A recent development in interacting particle systems and random walks is the extension of these theories to hypergraphs [LP12, CD12, CT13, CFR13, ALL14] and simplicial complexes [SKM14, PRT15]. We contribute to this direction by studying hypergraph analogues of coalescing/annihilating random walks and the voter model.

Besides the obvious fundamental interest of such a generalization, the models we consider arose naturally from an application, the analysis of a local search procedure called WalkSAT for solving instances of the satisfiability problem, in the special case of the so-called XOR-SAT version of SAT. It has other applications, unrelated at first sight to the first one: the theory of social balance [AKR06] and that of lights-out games [Sch14]. On the other hand the study of these systems, though it preserves some properties from the graph case has additional interesting features: for instance for so-called annihilating random walks on hypergraphs the number of particles is **not** in general nondecreasing (as it is in the graph case) and the structure of recurrent states is interestingly constrained by systems of linear equations similar to the ones used to analyze lights-out games [Sch14]. On the other hand, in coalescing random walks on hypergraphs there may be more than one copy of an initial "ball" and the process is naturally described using *multisets* rather than sets of balls.

The plan of the paper is as follows: first we define the models we are interested in and outline their motivation. In Section B. we present the (still open in general) issue of reachability and recurrence for annihilating random walks, together with a result settling this for our intended applications. In such a setting, our main result (Theorem 5.1 in Section 5.) upper bounds expected annihilation time in terms of a Cheeger-like constant of the hypergraph. We conclude with an application of this result to the analysis of the running time of the WalkSAT algorithm on instances of k-XOR-SAT and other (brief) remarks.

2. PRELIMINARIES

Hypergraphs considered in this paper are *simple*: for every two hyperedges e, f, $|e \cap f| \leq 1$. On the other hand we will allow *self-loops*, i.e. hyperedges e with $|e| = 1$. We will even allow multiple self-loops to the same vertex. A *multiset* is a set whose elements have a (positive) multiplicity. The *disjoint union* of multisets A and B, denoted $A \sqcup B$, is the multiset that adds up multiplicities of an element in A and B.

Given hypergraph $H = (V, E)$ and $v \in V$ we will denote by $N(v)$ its *open neighborhood*, defined as the set $\{w \neq v \in V : (\exists e \in E), \{v, w\} \subseteq E\}$ and by $N[v] = \{v\} \cup N(v)$ its *closed neigborhood*.

A. Setting and Related Work

In this paper we are concerned with a version of satisfiability called k-XORSAT:

Definition. *Given constant $k \geq 2$, an instance of k-XORSAT is a linear system of equations $A \cdot \overrightarrow{x} = \overrightarrow{b}$ over boolean field $\mathbf{Z}_2$, where A is an $m \times n$ matrix, for some $m, n \geq 1$, $\overrightarrow{x} = (x_1, x_2, \ldots, x_n)^T$ is an $n \times 1$ vector, $\overrightarrow{b} = (b_1, b_2, \ldots, b_m)^T$ is an $m \times 1$ vector, and each equation has exactly k variables.*

Though k-XORSAT can be solved in polynomial time by Gaussian elimination, we will not be concerned with this algorithm. Instead we aim to analyze a local search procedure, the WalkSAT algorithm [Pap91], displayed in Figure 2..1 and originally investigated on random instances of k-SAT. Though possible in principle in several cases (e.g. [Sch99],[ABS06], [COFF$^+$09], [COF14], [Zho13]) and well-understood from the standpoint of Statistical Mechanics [SM03, SM04], such an analysis is still quite complicated in general.

Here we trade off the difficulty of studying WalkSAT on random instances of k-SAT for the simpler problem k-XORSAT. While the study of such a scenario on random instances is still interesting [BHW03, SM03, AMZ08], interestingly, we obtain rigorous upper bounds on the expected running time of WalkSAT on *individual* solvable instances in terms of measurable parameters of *these individual instances.*

Algorithm 2..1: ALGORITHM WALKSAT(Φ)

Start with an arbitrary assignment U
while (there exists some unsatisfied clause)
  pick a random unsatisfied clause C
  change the value of a random variable of C in U.
return assignment U.

A second motivation comes from the physics of complex systems and is given by the following dynamics, first investigated by Antal, Krapivsky and Redner:

Definition. Constrained Triadic Dynamics*[AKR06, Ist09]. We start with a graph $G = (V, E)$ whose edges are labeled $0/1$. A triangle T is G is called* balanced *if the sum of the labels of its edges is 0 (modulo 2). At any step t, we chose an imbalanced triangle T uniformly at random and we change the sign of a random edge of T (thus making T balanced). The move might, however, make other triangles unbalanced.*

CTD can be modeled by the WalkSAT algorithm on an instance of 3-XORSAT [RVYMO06]. As further shown in [Ist09], one can sometimes analyze CTD using duality.

3. MOTIVATING EXAMPLE: THE CASE OF A COMPLETE 5-UNIFORM LINEAR SYSTEM.

To motivate some of the concepts we introduce, in this section we study a particular instance of our problem, the *complete 5-uniform linear system*, defined as follows:

Definition. *Let $n \geq 5$ and let $Z \in \{0,1\}^n$ be a boolean vector. The* complete 5-uniform linear system $K_5(Z)$ *is the linear system with n boolean variables $X_1, X_2, \ldots, X_n$ and equations*

$$\sum_{i \in A} X_i = \sum_{i \in A} Z_i$$

where A ranges over all 5-element subsets of $\{1, 2, \ldots, n\}$.

By design $K_5(Z)$ has Z among the solutions. The following is an easy observation:

Lemma 3.1. *Z is the only solution of $K_5(Z)$.*

Proof. Subtracting two equations we infer $X_i - X_j = Z_i - Z_j$ for all $1 \leq i, j \leq n$. Thus the values of all variables are determined by the value of X_1.

When $X_1 = Z_1$ we obtain solution Z. The alternative $X_1 = Z_1 + 1$ does not lead to a solution, because it corresponds to flipping all bits in Z_1, which is not a solution of the system (as all equations have odd width). ■

From Lemma 3.1 to any assignment U_t considered at step t by $WalkSAT$ one can associate a partition $(A_t, \overline{A_t})$ of the variables $\{X_1, X_2, \ldots, X_n\}$ with

$$A_t = \{X_i : U_t(X_i) \neq Z_i\}, \text{ the set of "bad variables".)}$$

Eventually, with probability $1 - o(1)$, we will have $A_t = \emptyset$ (that is WalkSAT will find assignment Z). This holds in the general setting of a satisfiable system:

Theorem 3.2. *Let Φ be a satisfiable instance of k-XOR-SAT. Let X_0 be an arbitrary assignment. Then a satisfying assignment X_2 for Φ is reachable from initial assignment X_1 by means of moves of the WalkSAT algorithm.*

Equivalently, by duality: let w_0 be the configuration in the hypergraph $D(\Phi)$ corresponding to X_0 and let w_2 be the "all-zeros" configuration. Then w_2 is reachable from w_0.

Proof. Let X_0 be an initial assignment. We will prove that a solution of the system is reachable from X_0 by induction on k, the Hamming distance between X_0 and solutions of the system $A \cdot x = b$ (we will assume that vector X is such a solution with $d_H(X_0, X) = k$).

- **Case** $k = 0$**.** Then $X_0 = X$ and there is nothing to prove.
- **Case** $k = 1$**.** Then X_0 and X differ on a single variable z. Let w be an equation containing z. Then X_0 does not satisfy w (as X, which only differs on z, does). Choosing equation w and variable z we reach X from X_0.
- **Case** $k \geq 2$. If there is an equation w not satisfied by X_0 (but satisfied by X) then w must contain a variable on which X_0 and X differ. Let z be such a variable. Then by flipping z one can reach from X_0 an assignment X_1 at Hamming distance $k-1$ from X. Now it is easily seen that system $H(X_1, X)$ has solutions: any solution of $H(X_0, X)$ with the value of z flipped. By the induction hypothesis one can reach a solution from X_1, therefore from X_0.

■

We analyze the WalkSAT algorithm on $K_5(Z)$ by investigating the dynamics of the potential function $H(t) = |A_t|$. Eventually w.h.p. $H(t) = 0$, and we would like to investigate the expected hitting time of function $H(t)$.

WalkSAT evolves by flipping the value of a single variable. Therefore H_t can either decrease by 1 (if one "bad" variable becomes "good") or increase by one (if one "good" variables flips to "bad"). The following easy observation is crucial:

Lemma 3.3. *For every $t \geq 0$ an equation e is* not *satisfied by assignment U_t if and only if*

$$|Var(e) \cap A_t| \text{ is odd.}$$

This lemma motivates the following "exotic" notion of "odd cut" in a hypergraph:

Definition. *Given hypergraph H and cut $A, \overline{A}$ define*

- *$OddCut(A, \overline{A})$ to be the subhypergraph of H induced by edges e such that $|e \cap A|$* **is odd.**
- *$E^-(A, \overline{A})$ to be the set of pairs (v, e) with $v \in A$ and $e \ni v$, $e \in OddCut(A, \overline{A})$.*
- *$E^+(A, \overline{A})$ to be the set of pairs (w, e) with $w \in \overline{A}$ and $e \ni w$, $e \in OddCut(A, \overline{A})$.*

Remark. In the definition of E^- we may allow (odd-size) edges that do not contain a single vertex from $\overline{A}$! For instance an edge of odd size all whose vertices are in A counts as an edge crossing the cut.

The connection between these notions and the analysis of WalkSAT is clear: $\Delta H(t) = -1$, precisely when at step t the chosen pair (v, e) belongs to $E^-(A_t, \overline{A_t}$. Similarly, $\Delta H(t) = -1$, precisely when at step t the chosen pair (v, e) belongs to $E^-(A_t, \overline{A_t})$.

Definition. *For a hypergraph H and set $A \subseteq V(H)$ define the* odd Cheeger drift *$D_{odd}(A)$ as*

$$D_{odd}(A) = \frac{|E^+(A, \overline{A})| - |E^-(A, \overline{A})|}{|E^+(A, \overline{A})| + |E^-(A, \overline{A})|},$$

Note that the odd Cheeger drift $D_{odd}(A)$ is only well-defined for sets A such that $OddCut(A) \neq \emptyset$.

The characterization of all hypergraphs for which $OddCut(A) \neq \emptyset$ is related to existing concepts concepts related to *parity domination in graphs* [Sut89, AS92]:

Definition. *Given graph $G = (V, E)$, set of vertices $\emptyset \neq A \subseteq V$ is* (closed) even dominating in G *if for every $v \in V$, $|A \cap N[v]|$ is even.*

The adaptation to hypergraphs is as follows:

Definition. *Given hypergraph $H = (V, E)$, set of vertices $\emptyset \neq A \subseteq V$ is* even parity dominating in H *if for every $e \in E$, $|A \cap e|$ is even.*

The previous adaptation does not directly extend to hypergraphs the usual definition of parity dominating set in graphs [AS92]. Instead, to any graph $G = (V, E)$ one can associate its *neighborhood hypergraph* $N(G) = (V_N, E_N)$ whose vertices are those in V (i.e. $V_N = V$), and whose hyperedges correspond to closed neighborhoods of vertices in V. In other words

$$(\overline{e} \in E_N) \Longleftrightarrow (\exists v \in V : \ \overline{e} = N[v])$$

One can easily verify that a set of vertices in a graph G is even closed parity dominating if and only if it is even parity dominating in $N(G)$ (in the sense of the previous definition).

The introduced terminology allows us to characterize hypergraphs H such that $D_{odd}(A)$ is well-defined for all $\emptyset \neq A \neq V$ by the absence of an even dominating set in H.

Definition. *We will call connected hypergraph H* odd connected *if it has no even dominating independent set.*

We now return to the odd Cheeger drift, presenting a couple of examples:

Example 1. *For every regular* graph G*, the odd Cheeger drift of an arbitrary set A is zero, as $|E^{+}(A,\overline{A})| = |E^{-}(A,\overline{A})|$ for all A.*

Example 2. *Let $H = K_{n,5}$ be the complete 5-uniform hypergraph with n vertices. Then the odd Cheeger drift of arbitrary low-density subsets of H will be negative (for large values of n). Indeed, if $|A| = \delta n$, the number of hyperedges containing*

- *five vertices in A is $\binom{\delta n}{5} \sim \frac{\delta^5}{5!} n^5$. Each vertex will count for $E^{-}(A,\overline{A})$.*
- *three vertices in A is $\binom{\delta n}{3}\binom{(1-\delta)n}{2} \sim n^5\delta^3(1-\delta)^2/12$. These three vertices will count for $E^{-}(A,\overline{A})$, the other two for $E^{+}(A,\overline{A})$.*
- *exactly one vertex in A is $\delta n\binom{(1-\delta)n}{4} \sim n^5\delta(1-\delta)^4/24$. This vertex will count for $E^{-}(A,\overline{A})$, all the rest for $E^{+}(A,\overline{A})$.*

Thus

$$|E^{-}(A,\overline{A})| \sim n^5[\frac{\delta^5}{24} + 3\frac{\delta^3\cdot(1-\delta)^2}{12} + \frac{\delta(1-\delta)^4}{24}]$$

On the other hand

$$|E^{+}(A,\overline{A})| \sim n^5[2\frac{\delta^3(1-\delta)^2}{24} + 4\frac{\delta(1-\delta)^4}{24}]$$

"Drift" quantity $D_{odd}(\delta) = \frac{|E^{-}(A,\overline{A})|-|E^{+}(A,\overline{A})|}{|E^{-}(A,\overline{A})|+|E^{+}(A,\overline{A})|} \sim \frac{\delta^5+4\delta^3(1-\delta)^2-3\delta(1-\delta)^4}{\delta^5+8\delta^3(1-\delta)^2+5\delta(1-\delta)^4}$ is plotted against density parameter δ in Figure 1.

The analysis of Example 3. allows us to finally settle the problem investigated in this section:

Theorem 3.4. *The expected convergence time of WalkSAT on system $K_5(Z)$ is exponential in n.*

Proof. A consequence of drift analysis. Formally, one can take n large enough such that $-1 < D_{odd}(A) < -0.1$ for all A with $0.1n < |A| < 0.2n$. We then apply the Simplified Drift Theorem from [OW08] and infer that there exists constants $c > 1 > d > 0$ such that

$$Pr[T < c^n] < d^n$$

Therefore the expected time to hit zero is at least $c^n(1 - d^n)$. ■

Remark. It is easily seen that in fact the theorem remains true for every system A such that

- A has an unique solution.
- There exist constants $0 < \eta_1 < \eta_2 < 1/2$ and $\epsilon > 0$ such that for every set $A \subset V$ with $\eta_1 n < |A| < \eta_2 n$, $D_{odd}(A) < -\epsilon$.

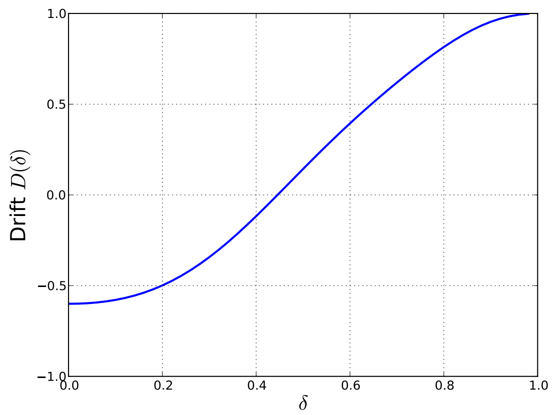

Fig. 1. The (asymptotic) odd Cheeger drift of the complete 5-uniform hypergraph $K_{n,5}$.

4. AN ANALYSIS OF THE GENERAL CASE VIA HYPERGRAPH DUALITY.

The previous section showed the relevance of concepts such as odd cuts and odd Cheeger drift in the analysis of the performance of algorithm WalkSAT.

A difficulty in extending the analysis to a general case is the lack of a good analog of the progress measure $H(t)$: when the system has multiple solutions the set of variables can no longer be partitioned into good and bad. Such a formula may still have a backbone or spine but frequently [IKKM12, AM15] the formula may have "minibackbones" corresponding to local clusters of solutions, but whose values differ between the exponentially many different clusters. Also, some variables may be outside the 2-core of the formula hypergraph, playing no role in its satisfiability, but be dependent on the variables in the 2-core (and possibly important in the dynamics of WalkSAT).

In the sequel we will take a different route: rather than concentrating on *variable-based measures* we will instead use *clause-based measure*. To this end we will analyze the dynamics of $H(t)$ the number of clause left unsatisfied by assignment U_t. Such an analysis will require us to consider structural properties of the *dual* of the formula hypergraph:

Definition. *Given instance* Φ *of* k*-XORSAT, the* dual $D(\Phi)$ *of* Φ *is an undirected hypergraph with self-loops* $D(\Phi) = (\overline{V}, \overline{E})$ *defined as follows:* $\overline{V}$ *is the set of equations of* Φ*. Hyperedges in* $D(\Phi)$ *correspond to variables in* Φ *and connect all equations containing a given variable. In particular we add a self-loop to an equation (vertex)* v *if it contains a variable appearing* only in v*. We may even add multiple self-loops to the same vertex.*

In other words $D(\Phi)$ is simply the dual of the formula hypergraph of Φ.

Note that if Φ is an instance of k-XORSAT then $D(\Phi)$ is a **k-regular hypergraph** (i.e. every vertex has degree exactly k).

Definition. *For any instance Φ of k-XORSAT let Z be an initial assignment to the variables of Φ. Let C_Z be the configuration of D_Φ defined as follows: a vertex v of C_Z has label 1 in C_Z if and only if the corresponding equation is satisfied by Z. C_Z will be called* the configuration dual to assignment Z.

We aim to upper bound the convergence time of WalkSAT on formula Φ in terms of Cheeger-like quantities of the dual hypergraph $D(\Phi)$. Obtaining such an analysis will turn out to require hypergraph analogues of annihilating and coalescing random walks, as well as of the voter model. Unlike their graph counterparts, and unlike the ordinary variants of random walks on hypergraphs [CFR13] or the recently defined *s-walks* [LP12], these models will be *explosive*: the number of "particles" may increase at a given step and be (in the case of coalescing random walk and voter model) unbounded. Such properties makes these models fundamentally different from their counterparts in the graph case.

A. Explosive random walks on hypergraphs

Motivating such explosive models starts by translating by duality the WalkSAT algorithm, performed as follows:

Proposition 4.1. *For any instance Φ of k-XORSAT let X_0 be an initial assignment to the variables of Φ. Let C_0 be the configuration dual to X_0.*

Suppose the algorithm WalkSAT on Φ with initial assignment X_0 changes variable x in (unsatisfied) clause C, resulting in assignment X_1. Then the dual configuration of X_1 in D_Φ is obtained by flipping the values of nodes in hyperedge x of D_Φ which contains node C whose initial value was 1.

Proof. By changing the value of variable x any equation that contains x and was satisfied by X_0 becomes unsatisfied by X_1 and viceversa. On the dual this reads as follows: every vertex of the hyperedge that corresponds to variable v changes value. ∎

This motivates the following definition:

Definition. *Let $H = (V, E)$ be a connected hypergraph. Define an* annihilating random walk on H *(Figure A.) by the following:*

(a). **Initial state:** Initially: $A_i \in \{0, 1\}$. We will identify this configuration with $\mathcal{B}$ the set of vertices $i \in V$ with $A_i = 1$, and call such a vertex i *live.*

(b). **Moves:** Choose pair i, e consisting of a **random live node** i and a random hyperedge $e = (i, j_1, \ldots, j_k)$ containing i. Simultaneously set $A_v = A_v \oplus A_i$ for all $v \in e$ (including $v = i$, which will result in $A_i = 0$).

It will be, however, more convenient to analyze annihilating random walks on k-uniform hypergraphs with two additional changes:

(a). first, we will study the *lazy* version of a.r.w., the one in which the choice of node i is not restricted to live nodes only.

(b). second, we will study annihilating random walks in *continuous*, rather than discrete, time.

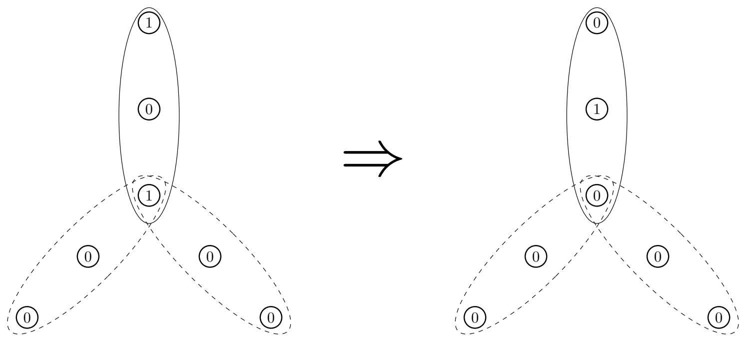

Fig. 2. One step of an annihilating random walk on hypergraphs.

Definition. [Lazy a.r.w. on hypergraphs]:

Let $H = (V, E)$ be a connected k-uniform hypergraph. Define a lazy annihilating random walk on H *by the following:*

(a). **Initial state:** *Initially: $A_i \in \{0, 1\}$. We will call a vertex i with $A_i = 1$* live.

(b). **Moves:** *Choose random node i and random edge $(i, j_1, \ldots, j_k)$ containing i. Simultaneously set $A_v = A_v \oplus A_i$ for all $v \in e$ (including $v = i$, which will result in $A_i = 0$).*

In discrete time the annihilation time of lazy a.r.w. provides, of course, only upper bound on annihilation in a.r.w (and, thus, convergence of the WalkSAT algorithm in the dual model). The introduction of continuous time does not create additional problems by the well-known equivalence between discrete and continuous time Markov processes with independent Poisson clocks.

Next we define an analogue of coalescing random walks for hypergraphs (Figure):

Definition. [(Lazy) coalescing random walks (c.r.w.) on hypergraphs]:

Let $H = (V, E)$ be a connected hypergraph. Each vertex holds a multiset of label A_i. Define a coalescing random walk on H *by the following:*

(a). **Initial state:** *$A_i \subseteq \{i\}$. Note that $\mathcal{B} := A_1 \cup A_2 \cup \ldots \cup A_n \subseteq [n]$. We will call a vertex i with $|A_i| = odd$* live.

(b). **Moves:** *Updating node i and hyperedge $e = (i, j_1, j_2, \ldots, j_k)$ (according to a Poisson clock) proceeds by making $A_{j_r} := A_{j_r} \uplus A_i$, for $r = 1, \ldots, k$ and $A_i = \emptyset$. Here $\uplus$ refers to the* **multiset sum**, *i.e. union with multiplicities. Note that the move never destroys any label (always $A_1 \cup A_2 \cup \ldots \cup A_n = [n]$) but may make some indices i satisfy $|A_i| = even$.*

(c). **Parity (coalescence) from $\mathcal{B}$:** *Given set of vertices $\mathcal{B} \subseteq V(G)$, $c_{coal}(H, \mathcal{B})$ is the minimum $t \geq 0$ such that, if starting with $A_v = \{v\}$ when $v \in \mathcal{B}$, $A_v = \emptyset$ otherwise, at time t $|A_j|$ is even for every j.*

Finally, the "dual" to coalescing random walks, a hypergraph analog of the voter model (Figure):

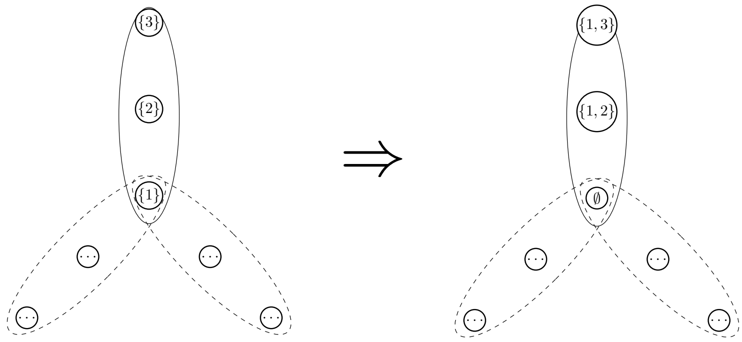

Fig. 3. One step of a coalescing random walk on hypergraphs.

Definition. [multiset voter model on hypergraphs]:

Let $H = (V, E)$ be a connected hypergraph. Define a multiset voter model on H *by the following:*

(a). **Initial state:** $A_i = \{i\}$. *Note that* $A_1 \cup A_2 \cup \ldots \cup A_n = [n]$. *W*

(b). **Moves:** *Updating node i and hyperedge $e = (i, j_1, j_2, \ldots, j_k)$ (chosen according to an independent Poisson clock) results in setting $A_i = \uplus_{r=1}^{k} A_{j_r}$. Note that the operation may decrease the number of different "opinions" present in the system, if such opinions were only held by node i.*

(c). **Parity of opinions:** *Given $\mathcal{B} \subseteq V(H)$,* parity time $c_{VM}(H, \mathcal{B})$ *is the minimum time t such that every initial opinion is present an even number of times (perhaps zero) among nodes in $\mathcal{B}$.*

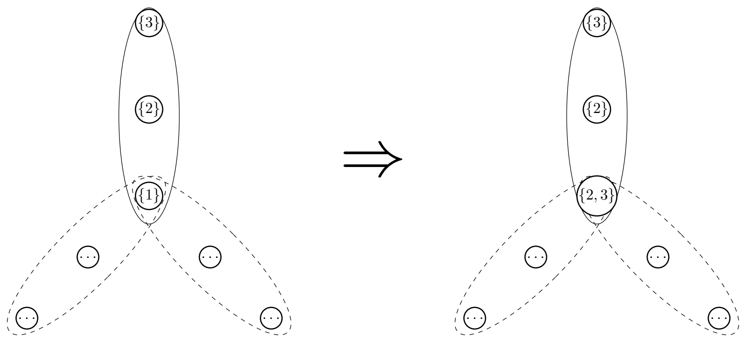

Fig. 4. One step of a multiset voter model on hypergraphs.

B. Annihilating random walks: reachability and recurrence

If the hypergraph H is actually a graph the long-term structure of configurations of the a.r.w. is simple: either a single live site survives (if $|V(H)|$ is odd) or none. In the general case the behavior is more complicated: the number of live nodes is **not** necessarily decreasing, as is the case in the graph setting. There may be, therefore, recurrent states different from **0** and those states with a single live node.

In the general case one can give [Ist09] a necessary condition for reachability:

Definition. *For every pair of boolean configurations* $w_1, w_2 : V(H) \to \mathbf{Z}_2$ *on hypergraph* H *we define a system of boolean linear equations* $H(w_1, w_2)$ *as follows: Define, for each hyperedge* e *a variable* z_e *with values in* $\mathbf{Z}_2$. *For any vertex* $v \in V(H)$ *we define the equation* $\sum_{v\in e} z_e = w_2(v) - w_1(v)$. *In the previous equation the difference on the right-hand side is taken in* $\mathbf{Z}_2$*; also, we allow empty sums on the left side. System* $H(w_1, w_2)$ *simply consists of all equations, for all* $v \in V(H)$.

Definition. *If* x *is a state on* H *and* l *is a hyperedge of* H, *define* $x^{(l)}(v) = 1 + x(v),$ *if* $v \in l$, $x(v)$, *otherwise.*

Lemma 4.2. *If state* w_2 *is reachable from* w_1 *then the system of equations* $H(w_1, w_2)$ *has a solution in* $\mathbf{Z}_2$.

Proof. Let P be a path from w_1 to w_2 and let z_e be the number of times edge e is used on path P (mod 2). Then $(z_e)_{e\in E}$ is a solution of system $H(w_1, w_2)$. Indeed, element $w(v)$ (viewed modulo 2) flips its value any time an edge containing v is scheduled.

■

In a previous paper [Ist09] we claimed a partial converse of Lemma 4.2. As the result below shows, though, the converse of Lemma 4.2 is however **not** true. We give two types of counterexamples. The first one is the setting in [Ist09]: connected hypergraphs without graph edges. Interestingly enough, for a modulo-p generalization of our dynamics (with $p \geq 3$) such a counterexample does not exist [Ist15]. The second counterexample shows that the failure of the converse implication is not specific to hypergraphs: even on graphs the sufficient condition fails to be necessary.

Theorem 4.3. *There exist*

(a). *a connected hypergraph* H *that contains no graph edges, and*
(b). *a connected graph (i.e. all hyperedges have size two)* H,

as well as two configurations w_1, w_2 *on* H *such that system* $H(w_1, w_2)$ *has solutions in* $\mathbf{Z}_2$, *yet* w_2 *is not reachable in* H *from* w_1.

Proof.

1. Let H be a hypergraph consisting of three hyperedges e_1, e_2, e_3 sharing a common vertex (Figure). Let w_1, w_2 be the configuration described in that figure: the private vertices of e_2 (displayed with a solid line in Figure) have initial value 1 in w_1, all other vertices being 0. On the other hand w_2 takes value 0 on the shared vertex and 1 everywhere else.

   It is easy to see that system $H(w_1, w_2)$ has a solution z with $z(e_1) = z(e_3) = 1$ and $z(e_2) = 0$. Yet w_2 is not reachable from w_1. Indeed hyperedges with three labels of one have no preimage. So the only preimages of state w_2 are itself and the three ones obtained by flipping labels on one hyperedge.

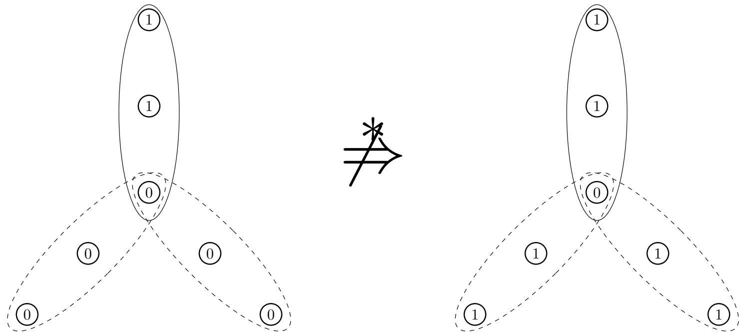

Fig. 5. Unreachability in a hypergraph with no graph edges.

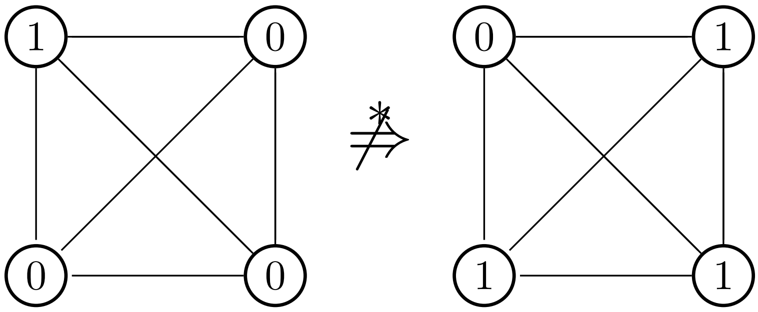

Fig. 6. Unreachability in a graph.

2. Let H be the complete graph K_4 and let w_1 be 1 at a single vertex v. Let w_2 be the configuration with ones at *every vertex but v*. System $H(w_1, w_2)$ has solution $z_e = 1$ for every edge e, yet w_2 is **not** reachable from w_1, as w_1 has a single one and w_2 has three, but on a graph the number of ones does not increase.

■

While we raise the complexity of reachability as an open problem, we believe it is possible to patch up the result in [Ist09] (perhaps by imposing meaningful restrictions on states w_1, w_2) and further extend it in order to provide a large class of reachability instances for which the necessary condition in Lemma 4.2 is also sufficient.

On the other case the case $w_2 = 0$, that corresponds to the setting we consider in this paper, that of a satisfiable XOR-formula

de continuat

C. Coupling annihilating and coalescing random walks

A particular setting we would like to investigate is given by our motivating examples: the XOR-SAT problem if the system has a solution and the CTD for social balance. With these cases in mind we define the analogue of annihilation time for annihilating random walks on hypergraphs:

Definition. [**Annihilation time:**]

Given set of vertices $\mathcal{B} \subseteq V(G)$, $c_{ann}(G, \mathcal{B})$ is the minimum $t \geq 0$ such that in the lazy a.r.w on G started with $A_i = 1$ if $i \in \mathcal{B}$, $A_i = 0$ otherwise, at time t we have $A_i = 0$ for all i.

When $E[c_{ann}(G, \mathcal{B})] < +\infty$ the associated configuration $\mathcal{B}$ will be called stabilizing.

We now extend a coupling argument valid in the case of graphs:

Theorem 4.4. *Suppose G is a connected hypergraph and $\mathcal{B} \subseteq V(G)$ is a stabilizing configuration. Then*

- *in the coalescing random walk on G starting from $\mathcal{B}$ one can reach coalescence.*
- *one can couple the coalescing and annihilating random walks on G such that $c_{ann}(G, \mathcal{B}) \leq c_{coal}(G, \mathcal{B})$.*

Proof. We will define the following stochastic process P:

1. **Initial state:** $A_i = \{(i, \infty)\}$ for $i \in \mathcal{B}$, $A_i = \emptyset$ otherwise. Note that $A_1 \cup A_2 \cup \ldots \cup A_n = \mathcal{B} \times \{\infty\}$ and that each A_i contains at most one index b_i with $(b_i, \infty) \in A_i$. We will call such a set *live* and b_i *the witness* for A_i. Also denote $B_i = A_i \setminus \{(i, \infty)\}$ if i is live, $B_i = A_i$ otherwise.
2. **Move:** At time t: Choose random vertex i (not necessarily live). Choose random edge $(i, j_1, \ldots, j_k)$. For $r = 1, \ldots, k$
   - If both A_i, A_{j_r} are live then make $A_{j_r} = (B_i \cup B_{j_r}) \cup \{(b_i, t), (b_{j_r}, t)\}$.
   - If, on the other hand, at most one of A_i, A_{j_r} is live then make $A_j := A_i \cup A_{j_r}$.

   Finally make $A_i = \emptyset$. Note that if we "move" a dead set A_i to a live set A_j then A_j will still be live.
3. **Stopping:** *Stopping time* $c_P(G)$ is the minimum $t \geq 0$ such that at most one i is live (one if n is odd, none if n is even)

Claim. *The following are true:*

1. P **observed on** $[n] \times \{\infty\}$ **and moves of live sets only** *is the annihilating random walk on G starting from configuration $\mathcal{B}$. If n is even then at time $c_P(G)$ all particles have annihilated. Consequently $c_{ann}(G, \mathcal{B}) \leq c_P(G, \mathcal{B})$.*
2. *P where we disregard second components in all pairs is identical to the coalescent random walk on G and $c_P(G, \mathcal{B}) = c_{coal}(G, \mathcal{B})$.*

A "proof by picture" is given in Figure 7. There are two cases: j is live or not. In both cases the observed process is identical to the annihilating random walk. Note that if n is even then when coalescence occurs in the c.r.w. all particles have died in the a.r.w.

∎

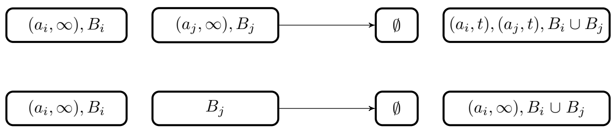

Fig. 7. The two cases of stochastic process P. Only two nodes inside a common hyperedge are pictured.

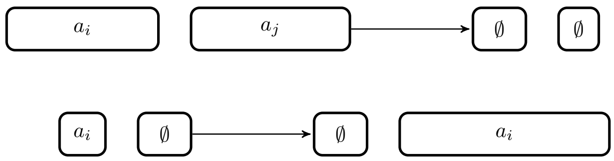

Fig. 8. First coupled version: annihilating random graphs (the two cases). Only two nodes inside a common hyperedge are pictured.

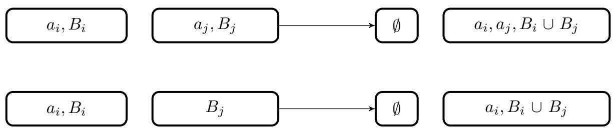

Fig. 9. Second coupled version: coalescing random walks (the two cases). Only two nodes inside a common hyperedge are pictured.

The reason a result such as Theorem 4.4 is interesting is that on graphs (see [AF14]) $c_{coal}(G)$ is identical (via duality) to coalescence time of voter model $c_{VM}(G)$, which can in turn be upper bounded in terms of a so-called *Cheeger time of graph* G, essentially the inverse of the more well-known Cheeger constant of G.

Similar results holds on hypergraphs, although we will need to give them in a slightly more general form:

Theorem 4.5. *For any hypergraph H and $\mathcal{B}$ is a stabilizing configuration. Then the following are true:*

- *one can reach parity on $\mathcal{B}$ in the multiset vector model.*
- *the coalescence time $c_{coal}(H, \mathcal{B})$ and the parity time of the associated multiset voter model $c_{VM}(H, \mathcal{B})$ are identically distributed.*

Proof. The proof is an adaptation of the classical duality argument [AF14]: we will define a process on *oriented hyperedges* in H (that is edges with a distinguished vertex) that will be interpreted in two different ways: as parity in the multiset voter model and coalescence in the coalescent random walk.

The process is described in Figure 10. There is a certain difficulty in correctly drawing pointed events in hypergraphs. In the figure we represent hyperedges vertically at the moment the given hyperedge event occurs (times t_1 and t_2 in the coalescing random walk), but this may be more difficult to draw if the vertices of a hyperedge are not contiguous.

Horizontal lines (e.g. for ball 3 between moments t_1 and t_2) refer to histories not interrupted by any hyperedge event between the corresponding times. A horizontal line may be interrupted by a hyperedge event. In the interest of readability we chose to drop some horizontal lines from the picture (e.g. at node 3 between time 0 and t_1).

A *left-right path* P between node i and node j is a sequence of hyperedge events and horizontal lines such that:

- P starts with a horizontal line of node i and ends with a horizontal line of node j.
- Every horizontal line of a node is followed by a hyperedge event with the corresponding node being pointed.
- Every hyperedge event is followed by an unique horizontal line corresponding to a **non-pointed node**.

For instance, in the picture from Figure 10 we have represented three left-right paths, between node 2 and each of nodes 1,4,5.

In the c.r.w. the activation of a hyperedge $e = [j \to i_1, i_2, \dots i_r]$ pointed at vertex j is interpreted as vertex j being chosen (together with edge e), thus sending a copy of its cluster of balls to all other neighbors.

In the multiset voter model the activation of a hyperedge $e = [j \to i_1, i_2, \dots i_r]$ pointed at vertex j is interpreted as j adopting the multiset union of opinions of $i_1, i_2, \dots, i_r$.

For instance, in the picture in Figure 10:

- in the c.r.w., assuming that initially $A_i = \{i\}$, $i = 1, 5$, at moment t_0 we have $A_1 = \{1, 2\}$, $A_2 = \emptyset$, $A_3 = \emptyset$, $A_4 = \{2, 3, 4\}$, $A_5 = \{2, 3, 5\}$.
- in the multiset voter model at moment t_0 we have $A_1 = \{1\}$, $A_2 = \{1, 4, 5\}$, $A_3 = \{4, 5\}$, $A_4 = \{4\}$, $A_5 = \{5\}$. Label 3 has disappeared from the system.

Just as in the ordinary c.r.w./voter model, the existence of a left-right path between nodes i and j (e.g. $(2, 1), (2, 4), (2, 5)$) is interpreted as the event:

- In the c.r.w.: "at time t_0 node j holds a ball with label i."
- In the multiset voter model: "at time t_0 node i holds opinion j with multiplicity at least one."

Moreover one path may contribute (when it does) with *exactly one ball/opinion* of a given type.

Consider now the event: "at t_0 every node in $\mathcal{B}$ on the right-hand side is connected to nodes on the left-hand side by an even number of paths".

- In the coalescing random walk this is equivalent to "at t_0 we have parity from $\mathcal{B}$"
- In the multiset voter model this is equivalent to "at t_0 we have parity of opinions on $\mathcal{B}$"

∎

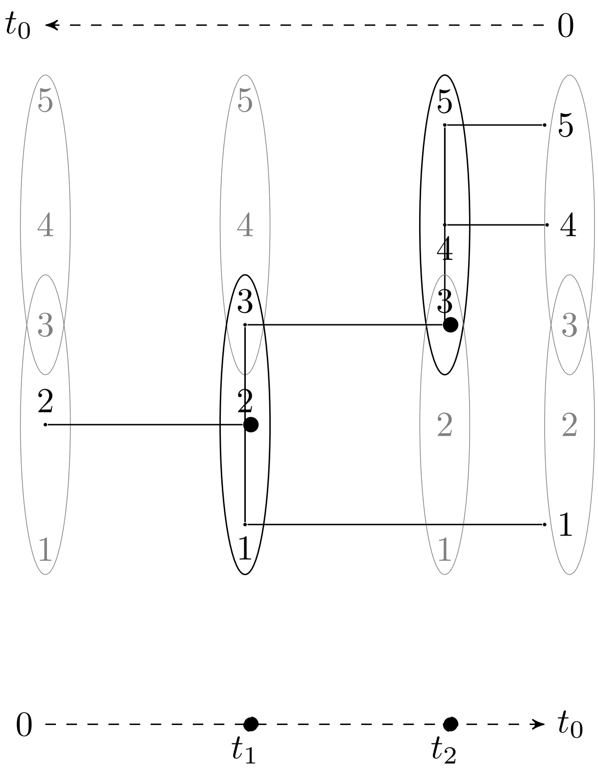

Fig. 10. Coupling the lazy coalescing random walk and the multiset voter model. Time runs from left to right in the lazy coalescing random walk and right to left in the multiset voter model. At time t_1 (in the lazy c.r.w.) copies of balls at (pointed) node 2 are sent to nodes 1 and 3. Similarly, at time t_2 copies of cluster at (pointed) node 3 are sent to nodes 4 and 5.

D. The multiset voter model and its two-party counterpart

Upper bounding the annihilation time of a.r.w. on graphs can be achieved [AF14] coupling the voter model with a "two-party" counterpart. In the sequel we accomplish a similar task on general hypergraphs:

Definition. [two-party voter model on hypergraphs]:
Let $H = (V, E)$ be a connected hypergraph. Define the two-party voter model on H *by the following:*

(a). **Initial state:** $A_i \in \{0, 1\}$ *for all* $i \in V$*. We denote* $\mathcal{A} = \{i \in V : A_i = 1\}$.

(b). **Moves:** *Updating node* i *and hyperedge* $e = (i, j_1, j_2, \ldots, j_k)$ *(chosen according to an independent Poisson clock) results in setting* $A_i = \oplus_{r=1}^{k} A_{j_r}$.

(c). **Parity:** *Given* $\mathcal{A}, \mathcal{B} \subseteq V(H)$*, the* parity time $c_{B-VM}(H; \mathcal{A}, \mathcal{B})$ *is the minimum time* t *such that, starting from configuration* $\mathcal{A}$*, at time* t *and subsequently*

$$\oplus_{i \in B} A_i = 0.$$

Note that, unlike the multiset voter model, in the two-party voter model we allow initial states $\mathcal{A}$ where at time $t = 0$ "some nodes do not hold any opinion".

Theorem 4.6. *Let* H *be a odd connected hypergraph and* $\mathcal{B} \subseteq V(H)$ *a stabilizing set. Then*

- *for every* $\mathcal{A} \subseteq V(H)$ *one can reach parity of opinions on* B *in the two-party voter model started from configuration* $\mathcal{A}$.
- *for every* $\mathcal{A} \subseteq V(H)$ *one can couple the multiset voter model and the two-state voter model with initial state* $\mathcal{A}$ *such that whenever we have parity on* $\mathcal{B}$ *in the multiset voter model we have parity of opinions on* $\mathcal{B}$ *in the two-party voter model started from configuration* $\mathcal{A}$.

Proof. Given a run of the multiset voter model, define a (coupled) run of the two-state voter model with initial state $\mathcal{A}$ by defining, for every $i \in V$ and every moment t, A_i to denote the parity of the multiset of opinions *from set* $\mathcal{A}$ *only* held at moment t by vertex i.

∎

Corollary. In the settings of Theorem 4.6 we have

$$Pr[c_{2-VM}(H; \mathcal{A}, \mathcal{B}) > t] \leq Pr[C_{VM}(H; \mathcal{B}) > t]$$

5. UPPERBOUNDING ANNIHILATION TIME IN A.R.W. ON HYPERGRAPHS

We first discuss and motivate our spectral measure, a variant of the Cheeger constant, of hypergraphs.

A. Spectral measures for hypergraphs

The classical definition of the Cheeger constant of a connected graph is [Chu97]:

$$\phi(G) = \min_{0<|A|\leq \frac{n}{2}} \frac{|E(A,\overline{A})|}{|A|},$$

where $E(A,\overline{A})$ is the *edge boundary of set* A, that is, the set of edges with one endpoint in set A and another one in its complement. In practice a related version is often used:

Definition. *The* edge expansion (or Cheeger constant) $h(G)$ of graph G *is defined as*

$$h(G) = \min_{0<|A|<n} \frac{n|E(A,\overline{A})|}{|A||\overline{A}|}, \tag{5.1}$$

On the other hand the analysis of voter model on graphs [AF14] required a different variant, related to the conductance of a graph and called in [AF14] *Cheeger time*:

Definition. *For a k-regular graph G define* the Cheeger time τ_G *of G by*

$$\tau_G = \sup_{0<|A|<n} \frac{k|A||\overline{A}|}{n\cdot |E(A,\overline{A})|} \Big(= \frac{k}{h(G)}\Big)$$

The definitions of Cheeger constant/time has recently been lifted up from graphs to simplicial complexes in several related but different ways:

- First, cohomological (coboundary) versions of Cheeger constant have appeared, implicitly or explicitly, in several works [Gro10, LM06, DK12, NR13]:

$$\phi(X) = \min_{f\in C^{k-1}[X,\mathbf{Z}_2]} \frac{|\delta_X f|}{||[f]||}$$

- On the other hand Parzanchevski et al.[PRT15] gave a combinatorial extension of the Cheeger constant to simplicial complexes:

$$h(X) = \min_{V=\sqcup_{i=1}^{k} A_i} \frac{n\cdot |F(A_1,A_2,\ldots,A_k)|}{|A_1|\cdot|A_2|\cdot\ldots\cdot|A_k|}$$

  where the minimum is taken over all partitions of V into nonempty subsets $A_1, A_2, \ldots A_k$ and by $F(A_1, A_2, \ldots, A_k))$ we have denoted the set of $(k-1)$-dimensional simplices with one vertex in each of $A_1, A_2, \ldots A_k$ (see also the slightly modified version $h'(X)$ of Gundert and Szedlák [GS15]).

Our desired connection with the voter model on hypergraphs will require, however, a specially tailored Cheeger-like quantity, somewhat similar to the definition of coboundary expansion but with an easy combinatorial definition:

Definition. *Given a k-uniform hypergraph H, define* the odd Cheeger time τ_H *as*

$$\tau_{odd}(H) = \sup_{0<|A|\leq|V|} \frac{nk}{|E^{-}(A,\overline{A})|},$$

The definition above coincides, in the case of connected graphs, with the Cheeger time of H defined in [AF14].

B. Main result

We can finally give our main result. It gives a bound useful only for odd connected hypergraphs H, as it requires that $\tau_{odd}(H) < +\infty$.

Theorem 5.1. *Given odd connected hypergraph H and set $\mathcal{B} \subseteq V(H)$ we have*

$$E[c_{VM}(H, \mathcal{B})] \leq T_H(n),$$

where $T_H(n)$ is a function such that:

1. $T_H(n) = O(n)$, *if there exists constants $\delta > 0$ such that for all nonempty set A, $D_{odd}(A) \geq \delta$.*
2. $T_H(n) = O(n^2 \tau_{odd}(H))$, *if case 1 does not apply but $D_{odd}(A) \geq 0$ for every nonempty set A.*
3. $T_H(n) = \Theta(e^{\Omega(n)})$, *if there exist $0 < \eta_1 < \eta_2 < |\mathcal{B}|/n$ and $\delta > 0$ such that for all sets $A \subset V$ with $\eta_1 n < |A| < \eta_2 n$ we have $D_{odd}(A) < -\delta$.*

Consequently similar upper bounds hold for the annihilating random walk on H.

A few comments on the bounds on $T_H(n)$ from the above result are in order:

- Our result can be restated as the claim that the odd Cheeger drift dictates the nature of the convergence time of the multiset voter model (and, with it, of the application to algorithm WalkSAT, see Subsection D. below). Of course, drift-based methods are well established in the analysis of local search heuristics [HY01, DJW02, LW14]. Our contribution can, therefore, be restated as identifying the odd Cheeger drift as the relevant quantity driving the dynamics of WalkSAT.
- Of course, conditions in Cases 1,2, 3 are not exhaustive. They are chosen by analogy with the possible situations in drift analysis (and, more generally, in that of mixing in Markov chains) with two polar limits, expansion, respectively a bottleneck cut.
- Case 1 is only included for completeness: we know of no hypergraph H to which it applies. Even a "variable (positive) drift" scenario [LW14] does not, we believe, occur in typical cases (see remark in Example B. below for an explanation of this intuition)
- The result arises from considering the evolution in time of N_t, the number of vertices having an odd number of opinions in the multiset voter model (or, equivalently of the coupled two-state voter model). In one update step N_t

can only go up/down by 1 or stay the same. On the other hand, for odd connected hypergraphs H, state **0** is the only absorbing state of the two-state voter model: for every configuration C different from **0** at least one edge e has an odd number of ones in C. Let v be a vertex in e labeled 1 in C. Scheduling pair (v, e) decreases the number of ones.

- In the first case of the Theorem the walk has a positive bias towards zero. The convergence time is therefore linear. In the second case the random walk is at worst unbiased. It will hit zero only as a result of diffusive behavior. The convergence time would be quadratic if the probability of moving to the left/right would be (lower bounded by) a constant. It is not in general, thus we need to slightly alter the result to take into account this phenomenon. The Cheeger constant appears in the final upper bound. Finally, in the last case, in the region $\eta_1 n < |A| < \eta_2 n$ the chain is driven, on the average, away from zero. With high probability it will eventually cross the barrier, but will need exponential time to do so.

Example 1. *Every graph G falls in the case 2 of the Theorem, and we fall over the result in [Ist09].*

Example 2. *As we have seen, the k-uniform complete hypergraph falls into Case 3 of the Theorem. Intuitively (but without proof) this should be the case for* **every** *hypergraph without graph edges. The reason is that when*

C. Proof of Theorem 5.1

Consider the following process, parameterized by a positive number $\epsilon > 0$, which yields a random model we will call $\mathbf{D}_\epsilon$:

- we partition the vertices of H into two parts, D and $\overline{D}$ by including each vertex into D independently with probability $1/2 - \epsilon$.
- we run the two-state voter model from configuration D (i.e. 0 on labels of vertices of $\overline{D}$ ("reds") and 1 on vertices of D ("blues")).
- We denote by D_t the set of vertices labeled 1 at time t, by $N_t^{D_\epsilon}$ the cardinal of D_t, and by $\Delta_t^{D_\epsilon}$ the difference in the number of ones as a result of the (possible) jump at time $t + dt$.
- Denote by C^{D_ϵ} the smallest time $t \geq 0$ when $N_t^{D_\epsilon} = 0$, where n is the number of vertices of H.

Denote by $c_{2-VM}^{\mathbf{D}_\epsilon}(H; \mathcal{B})$ the corresponding parity time.

Lemma 5.2. *For every* $0 < \epsilon < 1/2$ *we have*

$$E[c_{VM}(H; \mathcal{B})] \leq \frac{2}{1 - 2\epsilon} \cdot E[C_{2-VM}^{\mathbf{D}_\epsilon}(H; \mathcal{B})]$$

Suppose at time t we **do not** have parity on $\mathcal{B}$. Let C_t be the resulting configuration. Let $(a_1, a_2, \ldots, a_n)$ and the vectors of label parities on $\mathcal{B}$ of all initial opinions. By our assumption there must be two different opinions v_1, v_2 whose number of copies in $C_t|_{\mathcal{B}}$ have different parities:

$$a_i \neq a_j \text{ (mod 2)}.$$

We next apply the following trivial lemma, which simply follows from the fact that the two vectors are distinct and each entry on which they differ is independently chosen in $\mathbf{D}$ with probability $1/2 - \epsilon$:

Lemma 5.3. *Conditional on being in state C_t,*

$$Prob[\sum_{k\in \mathbf{D}} a_k \equiv 1 \textit{ (mod 2)}] \geq 1/2 - \epsilon.$$

Proof. If a_i is odd then including/excluding i from D changes the parity of $\sum_{j\in \mathbf{D}} a_j$. ∎

As a consequence we infer, similarly to the graph case [AF14], that

$$Prob[c_{2-VM}^{\mathbf{D}_\epsilon}(H;\mathcal{B}) > t] \geq \big(\frac{1}{2} - \epsilon\big) Prob[c_{VM}(H;\mathcal{B}) > t].$$

Finally, since for any random variable X on the nonnegative integers $E[X] = \sum_{i\geq 0} Pr[X \geq i]$, we infer

$$E[c_{VM}(H;\mathcal{B})] \leq \frac{2}{1-2\epsilon} \cdot E[C_{2-VM}^{\mathbf{D}_\epsilon}(H;\mathcal{B})]$$

Therefore all it remains now, to prove Theorem 5.1, is to show that

Lemma 5.4. *We have*

$$max_{\mathcal{B}} E[C_{2-VM}^{\mathbf{D}_\epsilon}(H;\mathcal{B})] \leq T_n(H)$$

for some function $T_n(H)$ with the properties from Theorem 5.1.
Proof.

First we prove the following result about the stochastic model $\mathbf{D}$:

Lemma 5.5. *We have*

$$Prob[\Delta N_t^D = -1] \geq \frac{1}{\tau_H} \cdot \frac{N_t^D \cdot (n - N_t^D)}{n}.$$

and

$$Prob[\Delta N_t^D = -1] - Prob[\Delta N_t^D = 1] = \frac{|E^-(D_t, \overline{D_t}| - |E^+(D_t, \overline{D_t}|}{nk}$$

Proof. N_t^D decreases by one exactly when the chosen vertex v has label 1 and the edge $e \ni v$ contains an odd number of nodes (including v !) with label 1. Similarly,

N_t^D increases by one precisely when the chosen vertex v has label 0 and the edge $e \ni v$ has an odd number of nodes with label 1.

The number of distinct vertex-edge pairs in the two-party multiset voter model is precisely kn, since every vertex of H has degree exactly k.

The number of vertex-edge pairs that lead to an increase by 1 is nothing but $|E^+(D_t, \overline{D_t})|$, with E^+ having the meaning from Definition 3. ■

We complete the proof of the upper bound as follows:

- In the first case we are in a constant (additive drift) scenario and we could apply basic results from drift analysis [HY01]. The following is a sketch of an independent proof:

  First we rewrite drift condition as

  $$E[\Delta N_t^D] = Prob[\Delta N_t^D = 1] - Prob[\Delta N_t^D = -1] \leq -\frac{\delta}{k}$$

  Because probabilities above belong to interval (0,1), $-1 \leq E[\Delta N_t^D] \leq -\frac{\delta}{k}$, hence $\delta \leq k$. Also

  $$\begin{aligned} Prob[\Delta N_t^D = 1] = \ & \tfrac{1}{2}[(Prob[\Delta N_t^D = 1] + Prob[\Delta N_t^D = -1]) + \\ & +(Prob[\Delta N_t^D = 1] - Prob[\Delta N_t^D = -1])] \leq \\ & \leq \tfrac{1}{2} - \tfrac{\delta}{2k} \end{aligned}$$

  also

  $$Prob[\Delta N_t^D = -1] \geq \ Prob[\Delta N_t^D = 1] + \tfrac{\delta}{k} \geq \tfrac{\delta}{k}$$

  In fact, as long as $N_t^D > 0$, ΔN_t^D can be stochastically upper bounded by a random variable X_t taking value 0 with the same probability as ΔN_t^D, and such that $E[X_t] = -\frac{\delta}{k}$.

  Since hyperedge choices are independent, random variables ΔN_t^D are also independent and we can take their dominating random variables X_t to be independent too.

  Given arbitrary independent random variables Z_t with values in $-1, 0, 1$ and $E[Z_t] = -\frac{\delta}{k}$ define chain (Y_t) by $Y_0 = Z_0$, $Y_t = Y_{t-1} + Z_t$. By standard application of elementary hitting time techniques (such as the forward equation and generating functions) to chain Y_t we infer

  $$E_{Y_0}[T_{\{Y_t=0\}}] = \frac{X_0 \delta}{k}.$$

  Applying this to chain N_t^D we infer

  $$\max_{\mathcal{B}} E[C_{2-VM}^{\mathbf{D}_\epsilon}(H; \mathcal{B})] \leq \frac{n\delta}{k}$$

- The argument is similar: we couple the process with a random walk Y_t on the integers with a reflecting barrier (upper bound) at n. We do so by requiring that for every $0 \leq k < n$,

$$Pr[\Delta N_t^D = 0 | N_t^D = k] = Pr[\Delta Y_t = 0 | Y_t = k]. \quad (5.2)$$

and redistributing the remaining probability equally between $Pr[\Delta Y_t = -1 | Y_t = k]$ and $Pr[\Delta Y_t = 1 | Y_t = k]$. From the hypothesis it follows that N_t^D can be stochastically dominated by Y_t, so upper bounds for the maximum hitting time of Y_t are upper bounds for the maximum hitting time of N_t^D as well.

By equality

$$Pr[\Delta Y_t = -1 | Y_t = k] = Pr[\Delta Y_t = 1 | Y_t = k] \geq \frac{1}{2} \cdot \frac{1}{\tau_{odd}(H)} \cdot \frac{k \cdot (n-k)}{n}.$$

Now we apply to chain Y_t Lemma 10 from [AF14]), comparing Y_t with the random walk on the integers, whose maximum hitting time is $\theta(n^2)$ The conclusion is that $T(n)$ can be taken to be $O(n^2 \cdot \tau_{odd}(H))$.

- Similar to the proof of Theorem 3.4.

■

D. Application to k-XOR-SAT

Putting the last three inequalities together, applying them to k-XOR-SAT and getting back from a continuous to an equivalent discrete time model we get an upper bound on convergence time of $WalkSAT$ on solvable instances H of k-XOR-SAT whose dual $D(H)$ is a simple hypergraph:

Corollary. Given instance Φ of k-XOR-SAT, $k \geq 3$, the following holds:

$$max_A E[WalkSAT(X_0)] \leq 2m^2 \cdot T_{D(H)}(m),$$

where m is the number of equations in H and the maximum is taken over all initial assignments X_0.

Proof. An easy consequence of the relation between discrete-time Markov chains and their continuous-time counterparts [Ald83], and the fact that $|V(D(H))| = m$. ■

6. CONCLUSIONS AND ACKNOWLEDGMENT

The work can be completed in many ways. Complete details and and many more results (for instance upper bounds on annihilation similar to those in [CEOR13]) should be a subject for further research. On the other hand, it would be interesting to see if the running time of other local search procedures, perhaps for more interesting problems like k-SAT can be analyzed in terms of (suitably defined) "particle systems".

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