Two notes on generalized Darboux properties and related features of additive functions

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Abstract - We present two results on generalized Darboux properties of additive real functions.

The first one introduces a weak continuity property, called **Q**-continuity, possessed by *all* additive functions. We show that every **Q**-continuous function is the uniform limit of a sequence of Darboux functions. The class of **Q**-continuous functions includes the class of Jensen convex functions. We discuss further connections with related concepts, such as **Q**-differentiability.

Next, given a **Q**-vector space $A \subseteq \mathbf{R}$ of cardinality $\mathbf{c}$ we consider the class $\mathcal{DH}^*(A)$ of additive functions such that for every interval $I \subseteq \mathbf{R}$, $f(I) = A$. We show that every function in class $\mathcal{DH}^*(A)$ can be written as the sum of a linear (additive continuous) function and an additive function with the Darboux property if and only if $A = \mathbf{R}$. We apply this result to obtain a relativization of a certain hierarchy of real functions to the class of additive functions.

Key words and phrases : additive functions, generalized Darboux properties, **Q**-continuity.

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1 Introduction

Structural properties (such as weak notions of continuity, Darboux property or Jensen convexity) of real functions, especially properties common to large classes of functions, have formed a significant part of the interests of professor Solomon Marcus in Real Analysis [26, 27, 28, 29, 30]. This paper presents two contributions in this area, motivated by the Darboux-like properties of additive functions. These functions are the solutions of the Cauchy functional equation

$$f(x + y) = f(x) + f(y)$$

(see [23] for a modern introduction). Together with researchers such as A.M. Bruckner, J. Ceder, F.B. Jones, W. Kulpa, F. Obreanu, J. Smítal, M. Weiss, Marcus investigated [26] structural properties of additive functions, such as the connectedness of the graph, and the Darboux property.

Besides the linear functions $f(x) = r \cdot x$ for some $r \in \mathbf{R}$, the class of additive functions contains a host of functions with "pathological" properties: nowhere continuity, nowhere monotonicity, a graph that is dense in $\mathbf{R} \times \mathbf{R}$. As noted in [23] (pp. 322) "discontinuous additive functions have many pathological properties. Therefore it is often believed that such functions cannot have any nice property." But this is not the case: since the writing of [17], a Ph.D. thesis [2] and several papers (among them [3, 1, 9, 10, 11, 35, 33]) have investigated Darboux-like properties and relaxations of continuity for additive functions.

The first contribution of this paper is the introduction of a weak continuity property, we call **Q**-continuity (here **Q** refers to the set of rationals) shared by *all* additive functions. We show that every **Q**-continuous function is the uniform limit of a sequence of Darboux functions. The class of **Q**-continuous functions includes the class of Jensen convex functions. We further discuss the related notion of **Q**-differentiability of a real function.

In the second note we consider the class $\mathcal{DH}^*(A)$ (where A is a vector subspace of $\mathbf{R}$ of cardinality c.) of additive functions such that for every interval $I \subseteq \mathbf{R}$, $f(I) = A$. We show that every function in class $\mathcal{DH}^*(A)$ can be written as the sum of a linear (additive continuous) function and an additive function with the Darboux property only if $A = \mathbf{R}$. The application of this result to the relativization of a function hierarchy to the class of additive functions is presented.

Results in this paper originate in my graduation thesis, written in 1994 under the supervision of Professor Marcus [17]. Rather than just publishing a set of old results, I have attempted, however to reconsider them from the vantage point of a more senior researcher (though one no longer actively working in Real Analysis): I have substantially modified my original definitions from [17], reproved some results and added new ones, and situated them in the context of more recent literature in Real Analysis. It is my hope (particularly with results in Section 2) that the notions presented here can stimulate further research.

The paper concludes with a recollection on my collaboration with professor Marcus and its students that led to these results.

2 Q-continuous functions and the inclusion $\mathcal{H} \subset \mathcal{U}$.

Let $\mathcal{H}$ be the class of additive functions, and let $\mathcal{U}$ be the class of functions that are the uniform limit of a sequence of Darboux functions. Bruckner, Ceder and Weiss [4] have observed that the inclusion $\mathcal{H} \subset \mathcal{U}$ is true, and follows from a result of Beckenbach and Bing [5] (every Jensen convex function is in $\mathcal{U}$). That is, the following hierarchy holds:

$$\mathcal{H} \subset \mathcal{J} \subset \mathcal{U}, \tag{1}$$

where $\mathcal{J}$ is the set of Jensen continuous function. On the other hand linear functions are, evidently continuous, and continuous functions have the Darboux property. We wondered whether these results "relativize" (under suitable weak notions of continuity and Darboux property) to all additive functions. That is, we wondered whether there exists a weak notion of continuity such that

(i). every additive function is "weakly continuous".

(ii). every "weakly continuous" function belongs to the class $\mathcal{U}$.

i.e. the inclusion $\mathcal{H} \subset \mathcal{U}$ is the consequence of some weak analogue of Darboux' Theorem ?

This question motivated work (presented below) in Section 5.4 our bachelor thesis [17]. Since then, the related question "are additive functions continuous in some weak sense" has been independently raised (according to [22]) by T. Salát.

In the sequel we present a weak continuity notion with properties (i). and (ii), somewhat modifying the solution from [17]:

Definition 2.1 *Let* $x \in \mathbf{R}$*. Function* $f : \mathbf{R} \to \mathbf{R}$ *is* $\mathbf{Q}$-continuous at x from the left *if there exists* $\epsilon > 0$ *such that for every* y, z *with* $x - \epsilon < y < z \leq x \in \mathbf{R}$*, there exists a continuous function* $\overline{f}_{y,z} : [y, z] \to \mathbf{R}$ *such that*

$$\overline{f}_{y,z} = f \text{ on the set } \{\alpha y + (1 - \alpha)z : \alpha \in \mathbf{Q} \cap [0, 1]\} \tag{2}$$

$\mathbf{Q}$-continuity at x from the right *is defined analogously.* $\mathbf{Q}$-continuity at x *is defined simply as* $\mathbf{Q}$*-continuity at* x *both from the left and from the right.*

We stress that in the above definition $\mathbf{Q}$ refers to the class of rational numbers. In particular our definition is to be distinguished from the (shorthand for the) class of quasicontinuous functions [36], often abbreviated in the literature as q-continuous.

Definition 2.1 should be compared to Banaszewski's $\mathcal{E}$-continuity [3], and to the similar notion of path Darboux property [25]. The former definition requires continuity via a system of paths, a feature shared (in a very restrictive form) by all $\mathbf{Q}$-continuous functions. However, our definition is substantially more "rigid": we need, in fact, an uncountable system of paths.

The following result provides a substantial number of examples of $\mathbf{Q}$-continuous functions:

Theorem 2.1 *The following are true:*

(a). *If* $f, g : \mathbf{R} \to \mathbf{R}$ *are functions that are (left, right, bilaterally)* $\mathbf{Q}$*-continuous at point* x_0 *then for all* $\alpha \in \mathbf{R}$*,* $f + g, \alpha \cdot f, f \cdot g$ *are (left, right, bilaterally)* $\mathbf{Q}$*-continuous at* x_0*.*

(b). *All additive functions* H *are* $\mathbf{Q}$*-continuous everywhere.*

Proof.

Immediate. For $f+g$ we take $\epsilon_{f+g} = \min\{\epsilon_f, \epsilon_g\}$ for the parameter in equation (2) and $\overline{f+g}_{y,z} = \overline{f}_{y,z} + \overline{g}_{y,z}$. For $f \cdot g$, given ϵ suitable for both f and g, for any $x - \epsilon \leq y < z \leq x$, by additivity the component functions f, g have continuous extensions $f_{y,z}, g_{y,z}$ with the properties in equation (2) on $[y,z]$. It is easy to see that $f_{y,z} \cdot g_{y,z}$ is such an extension for h.

The second part is trivial, since for $\alpha \in \mathbf{Q} \cap [0,1]$, $H(alpha \cdot y + (1-\alpha)z) = alpha \cdot H(y) + (1-\alpha)H(z)$. □

Lemma 2.1 *If $f : \mathbf{R} \to \mathbf{R}$ is (lower/upper/bilaterally) $\mathbf{Q}$-continuous at x_0 and $f(x) = 0$ locally (below/above/around) x_0, then $f(x_0) = 0$.*

Proof. Clearly $f_{y,z} = 0$ for close enough points y, z. Just choose a pair such that $x_0 \in \{\alpha y + (1-\alpha)z : \alpha \in \mathbf{Q} \cap [0,1]\}$. □

It will be also useful to introduce a weaker version of the class $\mathcal{U}$, denoted $\mathcal{U}_0$ (see [12, 24, 4]) that admits a "local" characterization.

Definition 2.2

1. *$\mathcal{U}_0$ is the class of functions $f : I \longrightarrow \mathbf{R}$ such that for every subinterval $J \subset I$, the set f(J) is dense in*

$$[inf_{x \in J}(f(x)), sup_{x \in J} f(x)].$$

2. *Define*

$$C_0^+(f,x) = \{y \in \overline{\mathbf{R}} | \forall M \in V(x_0), \forall z > y, card(f^{-1}(M) \cap (y,z]) \neq \emptyset\}.$$

Similarly (but working with intervals upper bounded by y) one defines the limit set $C_0^-(f,x)$.

3. *$f : I \longrightarrow \mathbf{R}$ is called* locally–$\mathcal{U}_0$ at point x *if the limit sets ${C_0}^-(f,x)$, ${C_0}^+(f,x)$ are intervals.*

Definition 2.3 *Let $I \subset \mathbf{R}$ be an interval. A function $f : I \to \mathbf{R}$ is* Jensen convex *if, for all $x, y \in I$,*

$$f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}.$$

The class of Jensen convex real functions will be denoted by $\mathcal{J}$. Clearly $\mathcal{U} \subset \mathcal{J}$.

The above-mentioned "local" characterization of class $\mathcal{U}_0$ is

Proposition 2.1 *[4] f belongs to the class $\mathcal{U}_0$ if and only if it is locally–$\mathcal{U}_0$ at every point x.*

With this definition we have the following result, that answers our question:

Theorem 2.2 *The following results hold:*

1. *Every Jensen convex function is* $\mathbf{Q}$*-continuous.*
2. *If* f *is* $\mathbf{Q}$*-continuous at* x*, then it is locally-*$\mathcal{U}_0$ *at* x*.*
3. *Every* $\mathbf{Q}$*-continuous function belongs to the class* $\mathcal{U}$*.*

In other words, we refine hierarchy 1 to:

$$\mathcal{H} \subset \mathcal{J} \subset \mathbf{QC} \subset \mathcal{U}, \tag{3}$$

Proof.

1. A result from [5] (see also comments in the proof of Theorem 4.5 from [4]) shows that condition (2) is true, if f is a Jensen convex function, for *every* $y < z$.

2. We will show that $C^-(f,x)$ is an interval (a similar result will hold for $C^+(f,x)$). Indeed, let $a < b \in C^-(f,x)$. Let $\xi \in (a,b)$ and consider two sequences x_n, y_n, $n \geq 1$, converging to x from below, such that

$$\lim_{n\to\infty} f(x_n) = a,$$

$$\lim_{n\to\infty} f(y_n) = b.$$

   Without loss of generality we may assume that $x_1 < y_1 < x_2 < y_2 < \ldots < x_n < y_n < \ldots < x$. Since f is $\mathbf{Q}$-continuous at x, for large enough n there exists continuous function $g_n :\overset{def}{=} g_{x_n,y_n} : [x_n, y_n] \to \mathbf{R}$ such that $f = g_n|_{\{\alpha x_n + (1-\alpha)y_n : \alpha \in \mathbf{Q}\cap[0,1]\}}$. Since g_n is continuous, for large enough n there exists $z_n \in [x_n, y_n] \cap \{\alpha x_n + (1-\alpha)y_n : \alpha \in \mathbf{Q} \cap [0,1]\}$ such that

$$|g_n(z_n) - \xi| \leq \frac{1}{n}.$$

   Since $f(z_n) = g_n(z_n) \to \xi$ (as $n \to \infty$), we infer that $C^-(f,x)$ is an interval.

3. Let us now consider an open interval N so that $f^{-1}(N) \neq \emptyset$ and $x_0 \in f^{-1}(N)$. Let $y_0 = f(x_0)$ and $\epsilon > 0$ so that $(y_0 - \epsilon, y_0 + \epsilon) \subset N$. For every $z \in \mathbf{R}$ there exists $\alpha \in \mathbf{Q}^*$ such that $|f(x_0 + \alpha z) - \xi| < \epsilon$. It follows that $x_0 + \alpha z \in f^{-1}(N)$, therefore $f^{-1}(N)$ is $\mathbf{c}$-dense in itself. By applying the characterization of class $\mathcal{U}$ (Theorem 3.2 in [4]) we infer that $f \in \mathcal{U}$.

$\square$

$\mathbf{Q}$-continuous functions have many other properties that are reminiscent of continuous functions. We give next a further example:

Definition 2.4 *Function* $f : \mathbf{R} \to \mathbf{R}$ *is* $\mathbf{Q}$-differentiable at x_0 *if there exists* $\lambda \in \mathbf{R}$ *such that function*

$$g_{x_0}(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & , \text{for } x \neq x_0 \\ \lambda & , \text{for } x = x_0, \end{cases}$$

is $\mathbf{Q}$*-continuous at* x_0*. The value* λ *is called* the $\mathbf{Q}$-derivative of f at x_0 *(we will write* $\lambda = f'_{\mathbf{Q}}(x_0)$*)*

Theorem 2.3 *If* f *is* $\mathbf{Q}$*-derivable at* x_0 *then* f *is* $\mathbf{Q}$*-continuous at* x_0*.*

Proof. We write the equation above as

$$f(x) = f(x_0) + (x - x_0) \cdot g_{x_0}(x)$$

and then use the fact that constants are $\mathbf{Q}$-continuous, $x - x_0$ is $\mathbf{Q}$-continuous and Lemma 2.1. □

Theorem 2.4 *The value* $f'_{\mathbf{Q}}(x_0)$ *of a function that is* $\mathbf{Q}$*-differentiable at* x_0 *is well defined (that is, there exists at most one such* λ*).*

Proof. Suppose there exist two completions λ, μ. By Lemma 2.1 the difference of the two functions g_{x_0} is a $\mathbf{Q}$-continuous that is locally zero around x_0. By Lemma 2.1 it must be that $\lambda = \mu$. □

Our extension of notions of continuity/differentiability is not as strong as one might believe: **no** additive function is $\mathbf{Q}$-differentiable, other than the linear functions.

Theorem 2.5 *Consider an additive function* f *that is* $\mathbf{Q}$*-differentiable at* $x_0 = 0$ *and let* $r = f'_{\mathbf{Q}}(x_0)$*. Then* $f(x) = rx$ *on an open neighborhood of zero.*

Proof. Consider x sufficiently close to zero. Then function g_0 is $\mathbf{Q}$-continuous at zero and constant on the set $\{\alpha y : \alpha \in \mathbf{Q}\}$. By the definition of $\mathbf{Q}$-continuity, with $y = x, z = 0$ we infer that $g_0(x) = r$, i.e. $f(x) = rx$ around zero. □

Kostyrko [22] has employed the terminology "type A property" to refer to continuity properties valid in the class of additive functions for linear functions only, and "type B" for properties that are not of type A. Extending this language one could say that $\mathbf{Q}$-differentiability of functions is a "locally type A property", while $\mathbf{Q}$-continuity is of type B.

One way to turn $\mathbf{Q}$-differentiability into a type B property is to relax Definition 2.4: rather than using as benchmarks for differentiability the linear functions, *we will use instead the set of all additive functions:*

Definition 2.5 *Given real function f and additive function H and $x_0 \in \mathbf{R}$, we say that function f is* $\mathbf{Q}$-differentiable at x_0 with respect to H *if there exists $\lambda \in \mathbf{R}$ and a function F_{x_0} with $F_{x_0}(x_0) = \lambda$, $\mathbf{Q}$-continuous at x_0 such that function*

$$f(x) = f(x_0) + [H(x) - H(x_0)] \cdot F_{x_0}(x).$$

The value λ is called the $\mathbf{Q}$-derivative of f at x_0 with respect to H *(we will write $\lambda = f'_{\mathbf{Q},H}(x_0)$)*

This change enables the beginnings of an (intriguing) differential calculus with respect to additive functions:

Theorem 2.6 *If functions f, g are $\mathbf{Q}$-differentiable at x_0 with respect to additive function H and $\alpha \in \mathbf{R}$, then $f + g, \alpha \cdot f$ and $f \cdot g$ are differentiable at x_0 with respect to H and*

$$(f+g)'_{\mathbf{Q},H}(x_0) = f'_{\mathbf{Q},H}(x_0) + g'_{\mathbf{Q},H}(x_0) \tag{4}$$

$$(\alpha f)'_{\mathbf{Q},H}(x_0) = \alpha \cdot f'_{\mathbf{Q},H}(x_0) \tag{5}$$

$$(fg)'_{\mathbf{Q},H}(x_0) = f(x_0) g'_{\mathbf{Q},H}(x_0) + g(x_0) f'_{\mathbf{Q},H}(x_0) \tag{6}$$

Proof. If

$$f(x) = f(x_0) + [H(x) - H(x_0)] \cdot F_{x_0}(x).$$

and

$$g(x) = g(x_0) + [H(x) - H(x_0)] \cdot G_{x_0}(x).$$

(where F_{x_0}, G_{x_0} are $\mathbf{Q}$-continuous at x_0, then

$$(f+g)(x) = (f+g)(x_0) + [H(x) - H(x_0)] \cdot (F_{x_0} + G_{x_0})(x)$$

and

$$(\alpha \cdot f)(x) = \alpha \cdot f(x_0) + [H(x) - H(x_0)] \cdot (\alpha \cdot F_{x_0})(x).$$

Finally

$$\begin{aligned}(fg)(x) - f(x_0)g(x_0) = [H(x) - H(x_0)] \cdot [&f(x_0)G_{x_0}(x) + f(x_0)G_{x_0}(x) \\ &+ [H(x) - H(x_0)] \cdot F_{x_0}(x) \cdot G_{x_0}(x)]\end{aligned} \tag{7}$$

so let

$$L(x) = f(x_0)G_{x_0}(x) + f(x_0)G_{x_0}(x) + [H(x) - H(x_0)] \cdot F_{x_0}(x) \cdot G_{x_0}(x)$$

We have

$$(fg)(x) - f(x_0)g(x_0) = [H(x) - H(x_0)] \cdot L(x)$$

and

$$L(x_0) = f(x_0) g'_{\mathbf{Q},H}(x_0) + g(x_0) f'_{\mathbf{Q},H}(x_0)$$

therefore relation (6) follows. □

Even definition (2.5) does not enlarge too much, though, the class of H-differentiable functions:

Theorem 2.7 *Given two additive functions the following are equivalent:*

1. *H_1 is **Q**-differentiable w.r.t. H_2.*
2. *There exists a constant r such that $H_1 = rH_2$.*

Proof. The reverse implication is easy. So let's deal with the direct one: suppose H_1 is differentiable w.r.t. H_2 at $x_0 = 0$, and define $r = (H_1)'_{\mathbf{Q},H_2}(0)$. We will show that $H_1 = r \cdot H_2$. From differentiability we infer

$$H_1(x) = H_2(x) \cdot g_0(x)$$

where function g_0 is **Q**-continuous at $x_0 = 0$. Consider, $x \in R$ (assume w.l.o.g. $x < 0$) such that $H_2(x) \neq 0$, and let $\alpha \in \mathbf{Q}$ such that relation (2) holds for function g_0 and parameters $y = \alpha \cdot x$, $z = 0$.

Since $H_1(\beta \cdot x) = \beta \cdot H_1(x)$ and similarly for H_2, we infer that for $\beta \in [0, \alpha] \cap \mathbf{Q}$ and continuous extension $\overline{g}_{[0,y]}$ of g_0

$$\overline{g}_{[0,y]}(\beta \cdot x) = \frac{H_1(x)}{H_2(x)} \tag{8}$$

Taking the limit $\beta \to 0, \beta \in \mathbf{Q}$ in equation (8) above we infer

$$\frac{H_1(x)}{H_2(x)} = (H_1)'_{\mathbf{Q},H_2}(0) = r.$$

□

3 Universally bad additive functions and the additive analogue of a hierarchy

In [18] we have studied the class $\mathcal{C} + \mathcal{D}$, of functions that are the sum of a continuous and a Darboux function, with the main purpose of understanding its relationship with the class $\mathcal{U}$, of functions that are the uniform limit of a sequence of Darboux functions. It is known that $\mathcal{C} + \mathcal{D} \subset \mathcal{U}$, and the inclusion is strict. It is also known [15] that $\mathcal{U}$ is closed under *quasi-uniform* (Arzelá-Gagaeff-Alexandrov) convergence, defined as follows:

Definition 3.1 *Let (X, ρ) and (Y, σ) be two metric spaces, and let $f, f_n : X \to Y$ be functions. Sequence $(f_n)_{n \geq 1}$ is said to* quasi-uniformly converge *to f iff*

(i) f_n converges pointwise to f.

(ii) For every $\epsilon > 0$ there exists a sequence (possibily finite) of indices $n_1 < n_2 < \ldots < n_p < \ldots$ and a corresponding sequences of open sets $G_1 < G_2 < \ldots < G_p < \ldots$ s.t. $X = \bigcup_i G_i$ and for all $i \geq 1$,

$$x \in G_i \Rightarrow \sigma(f(x), f_{n_i}(x)) < \epsilon.$$

We will employ notation $f_n \overset{AGA}{\to} f$.

Hence $\mathcal{U}$ it is also the closure of $\mathcal{D}$ under this type of convergence. If we denote by $U \cdot \mathcal{A}, QU \cdot \mathcal{A}$ the closure of a class $\mathcal{A}$ under uniform(quasiuniform) convergence, the above mentioned result reads

$$\mathcal{C} + \mathcal{D} \subset \mathcal{U} = U \cdot \mathcal{D} = QU \cdot \mathcal{D}. \tag{9}$$

An interesting variation on these classes considers an additional restriction on the functions involved: being *additive*. We will denote by $\mathcal{H}$ the class of additive functions and, for a class of functions $\mathcal{A}$, by $\mathcal{AH}$ the class $\mathcal{A} \cap \mathcal{H}$.

In the sequel we study the analogue of hierarchy (9) when the extra constraint of additivity is imposed. The resulting hierarchy does *not* simply mirror (9), if only for the following reason: as we have seen in the previous section, $\mathcal{H} \subset \mathcal{U}$, so in fact $\mathcal{UH} = \mathcal{H}$. On the other hand this class is easily seen *not* to be equal to $U \cdot \mathcal{DH}$ (since uniform convergence of additive functions is trivial). Finally, the comparison with $QU \cdot \mathcal{DH}$ is nontrivial.

Definition 3.2 *For $A \subset \mathbf{R}$ denote by $\mathcal{D}^*(A)$ the class of functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that for every interval $I \subset \mathbf{R}$, $f(I) = A$.*

For instance $\mathcal{DH} = \mathcal{D}^*(\mathbf{R})$. Indeed, the graph of every additive function is dense in $\mathbf{R} \times \mathbf{R}$. If it has the Darboux property then it must take every possible value in every interval.

We provide below examples of "universally bad" functions in the class of additive functions, in a manner similar in spirit to results in [1],[18], [34]:

Theorem 3.1 *If $A \subsetneq \mathbf{R}$ is a vector space with $|A| = \mathbf{c}$ then $\emptyset \neq \mathcal{DH}^*(A) \not\subset \mathcal{CH} + \mathcal{DH}$.*

Proof.

Let $\{r_\alpha\}_{\alpha < \mathbf{c}}$ be an enumeration of $\mathbf{R}^*$, where α ranges over all countable ordinals, and $g_\alpha(x) = r_\alpha \cdot x \in \mathcal{CH}$ be an enumeration of nontrivial linear functions. Finally, let $\{U_i\}_{i<\omega}$ be a basis for the usual topology on $\mathbf{R}$.

Let $H = \{h_\alpha\}_{\alpha < \mathbf{c}}$ be a Hamel base, dense in $\mathbf{R}$. The existence of such a base follows easily: starting from a Hamel basis H_1, if H_1 is not already dense then we shift its terms by rational numbers so that at least one term falls into each U_i.

Let $B = \{x_\alpha\}_{\alpha < \mathbf{c}}$ be a basis for A (it is clear that $|B| = \mathbf{c}$).

Define inductively sequences $y_\alpha \in \mathbf{R}, z_\alpha \in B, t_{\alpha,i} \in H \cap U_i, \alpha < \mathbf{c}, i < \omega$ so that

$$t_{\alpha,i} = t_{\gamma,j} \Rightarrow (\alpha, i) = (\gamma, j) \tag{10}$$

$$y_\alpha \notin < x_\beta - g_\alpha(t_{\beta,i}), z_\beta - g_\alpha(h_\beta) >_{\beta,i} \tag{11}$$

The construction is presented in the transfinite algorithm below.

Algorithm 3.1: CONSTRUCTION OF $(y_\alpha, z_\alpha, t_{\alpha,i})$

Assume we have defined $t_{\beta,i}, z_\beta, y_\beta, \forall \beta < \alpha, \forall i < \omega$.

The construction of y_α:

Since the **Q**-vector space $< x_\beta - g_\alpha(t_{\beta,i}), z_\beta - g_\alpha(h_\beta) >_{\beta<\alpha, i<\omega}$ is countable, we can chose y_α outside this space.

The construction of $t_{\alpha,j}, (j < \omega)$:

Suppose we have defined $t_{\alpha,k}, \forall k < j$.

The set $A_j = \{t \in H \cap U_j | x_\alpha - g_\alpha(t) \in$
$< x_\beta - g_\alpha(t_{\beta,k}), z_\beta - g_\alpha(h_\beta), y_\beta, z_\beta, y_\alpha, x_\alpha - g_\alpha(t_{\alpha,i}) >_{\beta<\alpha, i<j, k<\omega}\}$
is at most countable (since the **Q**-vector space in its definition is at most countable as well)
therefore we can choose $t_{\alpha,j} \in (H \cap U_j) \backslash A_j$.

The construction of z_α:

The set $B_\alpha := \{z \in B | z - g_\alpha(h_\beta) \in$
$< x_\eta - g_\alpha(t_{\eta,k}), z_\beta - g_\alpha(h_\beta), y_\eta, z_\beta >_{k<\omega, \eta \le \alpha, \beta < \alpha}\}$
is at most countable, therefore (since B has cardinal **c**) there exists $z_\alpha \in B \backslash B_\alpha$.

(10) results from our choice of $t_{\alpha,i}$.

To prove (11), assume y_α is a finite linear combination of elements from the vector space on the right side of (11).

(i). if all elements on the right-hand side of (11) have rank less that α we obtain a contradiction with our choice of y_α.

(ii). otherwise, let $\gamma > \alpha$ be the maximal rank in the linear combination. Considering the maximal rank term (one of $z_\gamma - g_\alpha(h_\gamma), x_\gamma - g_\alpha(t_{\gamma,i}),$ for some $i < \omega$) in the linear decomposition defining y_α.

If it is the first term, then by turning the inequality around, we can obtain $z_\gamma - g_\alpha(h_\gamma)$ as a finite linear combination with rational coefficients of y_α, $x_\gamma - g_\alpha(t_{\gamma,i})$ and lower order terms on the right-hand side of (11). But this contradicts the way we defined z_γ.

Suppose term $z_\gamma - g_\alpha(h_\gamma)$ does not appear in the finite linear combination defining y_α and, instead, the maximum rank term is $x_\gamma - g_\alpha(t_{\gamma,i})$ (for some i). Turning the inequality around we write $x_\gamma - g_\alpha(t_{\gamma,i})$ as a finite linear combination with rational coefficients of y_α and lower order terms on the right-hand side of (11). But this contradicts the way we defined $t_{\gamma,i}$.

Define $h : H \to \mathbf{R}$ by

$$h(x) = \begin{cases} x_\beta, & \text{if } x = t_{\beta,i} \text{ with } \beta < \mathbf{c}, i < \omega \\ z_\beta, & \text{if } x = h_\beta \notin \{t_{\gamma,i} | \gamma < \mathbf{c}, i < \omega\} \end{cases} \tag{12}$$

$h(H) = B$ and h is not injective. Extending by linearity h to all of $\mathbf{R}$, we obtain a function $h \in \mathcal{D}^*\mathcal{H}(A)$.

Supppose that $h = f_1 + f_2$ with $f_1 \in \mathcal{CH}$, $f_2 \in \mathcal{DH}$.

Case 1:

$f_1 \equiv 0 \Rightarrow h \in \mathcal{DH}$, which is impossible, since $A \neq \mathbf{R}$.

Case 2:

$f_1 = g_\alpha$ with $\alpha < \mathbf{c}$. Then $f_2(x) = h(x) - g_\alpha(x)$,

$$f_2|_H(x) = \begin{cases} x_\beta - g_\alpha(t_{\beta,i}), & \text{if } x = t_{\beta,i} \text{ for some } \beta < \mathbf{c}, i < \omega \\ z_\beta - g_\alpha(h_\beta), & \text{if } x = h_\beta \notin \{t_{\gamma,i} | \gamma < \mathbf{c}, i < \omega\} \end{cases} \tag{13}$$

Since $y_\alpha \notin < x_\beta - g_\alpha(t_{\beta,i}), z_\beta - g_\alpha(h_\beta) >_{\beta,i}$, it follows that $y_\alpha \notin range(f_2)$, therefore $f_2 \notin \mathcal{DH}$, a contradiction.

In conclusion, $h \in \mathcal{D}^*\mathcal{H}(A) \backslash (\mathcal{CH} + \mathcal{DH})$. □

Corollary 3.1 *The inclusion $\mathcal{CH} + \mathcal{DH} \subset \mathcal{H}$ is strict.*

Putting all things together we obtain a different hierarchy for the additive case:

Corollary 3.2 *We have*

$$\mathcal{DH} = U \cdot \mathcal{DH} \subseteq QU \cdot \mathcal{DH} \subseteq \mathcal{CH} + \mathcal{DH} \subset \mathcal{H}.$$

Proof. We prove that the quasiuniform limit of a sequence of additive Darboux functions is in $\mathcal{CH} + \mathcal{DH}$, then we apply Corollary 3.1.

Let $\{g_n\}_{n=1}^\infty$, $g_n \in \mathcal{DH}$, be a sequence of functions quasiuniformly convergent towards g. Clearly $g \in \mathcal{H}$. Let $\epsilon > 0$ and consider the neighborhood U of 0 and the index $n_\epsilon \geq 1$ s.t. $|g(x) - g_{n_\epsilon}(x)| < \epsilon, \forall x \in U$. Since $g - g_{n_\epsilon}$ is an additive function bounded in a neighbourhood of 0, it is a continuous function, therefore $g = (g - g_{n_\epsilon}) + g_{n_\epsilon} \in \mathcal{CH} + \mathcal{DH}$. □

4 A personal recollection

If memory serves me well I first met professor Solomon Marcus in the spring of 1989. I was 19, in that year between high-school and college when one had to complete nine-months military service, mandatory in communist Romania. The preceding year, still in high school, in a period of significant personal growth (I ranked first that year in the National Mathematical Olympiad in my age category), I became aware of the impending transition to a professional mathematician's career and started wondering what my (mathematical) future will be.

I was, at that time, somewhat aware of the basics of college-level mathematical education. Through my father, a high school mathematics teacher, I had access to most of the standard college-level textbooks in use at the University of Bucharest. I (thought I) was reasonably well acquainted with first-year calculus and had been attempting to become familiar with the basics of complex analysis and its applications to analytic number theory.

It wasn't meant for me to continue on that path. My scientific beginnings lie solidly in Professor Marcus's range of scientific interests: While still in highschool I had read professor Marcus's books [31] and [32]. I was struck by the (somewhat quaint nowadays) elegance and charm of the exotic properties in the theory of real functions, of the type investigated by Waclaw Sierpinski or Andrew Bruckner, in Romania by Simion Stoilow, Alexandru Froda and Professor Marcus himself. I became interested in generalizations of the intermediate value (a.k.a. Darboux) property, and sent professor Marcus a letter essentially containing what was to become later [15] one of my first scientific papers published (in 1991/92) outside Romania.

More importantly, during the summer of 1989 I had made the (crucial, for my career) connection to theoretical computer science: by sheer hazard, in the summer of 1988 I had stumbled (in an used bookstore in my hometown, Galaţi) upon two research-level monographs on computability theory and formal languages written by two of Professor Marcus's disciples: Cris Calude [6] and Gheorghe Pãun [37], and (thought I had) solved one open problem from each textbook. By the time I entered college I was the author of two scientific papers in theoretical computer science [13, 14] and the basic direction of my scientific path had pretty much been decided.

When I entered the University of Bucharest in the fall of 1989 I was **not** a student of professor Marcus: I simply was assigned to a different series which, on the other hand, had Cris Calude as professor. I continued to work in the research group of Marcus, Calude, Pãun, realizing soon enough that I was more interested in the research I was doing "on the side" than in the (fairly different) courses I was taking in the Faculty of Mathematics.

Communism fell, and with its fall came the possibility of unhindered publication outside Romania: Until 1994, the year I had left to pursue a

Ph.D. in the United States I was an author of several papers [7, 8, 16, 20]. More importantly, with it came the possibility to working abroad. This had an unfortunate impact on composition of the research group I belonged to: in 1992 Cris left for New Zealand. Marius Zimand, one of the members of the Marcus group and coauthor on paper [8], became in 1993 a Ph.D. student in Computer Science at the University of Rochester. I followed him in the fall of 1994, among the first in what later became a mass exodus of Romanian scientists.

As I was completing my undergraduate studies at the University of Bucharest I wrote a thesis [17] on the theory of real functions under the guidance of Professor Marcus. Its purpose was to create a survey (based on the literature available in pre-internet 1994 Romania) of the theory of generalized versions of the Darboux property. More importantly, it contained some original results, some of them (eventually) published in Real Analysis Exchange [15, 18]. A few of had, however, remained confined to my thesis.

More than 15 years after my college graduation (as Professor Marcus visited Timişoara as an invited speaker of SYNASC 2010) I was shocked to find out that he still remembered that some of the original results in my thesis had not been published. He suggested that I should revisit them, and eventually submit them for publication.

The purpose of this paper is to graciously answer this request. It is fitting to dedicate them to a man that has influenced my career in so many profound ways. Professor Marcus has undoubtedly had many students and disciples. While my American experience led my research career on a set of paths very different from those of my scientific beginnings (a recent sample is [21]), I would like to point out that I am probably one of the few students of professor Marcus to have worked in several of his major research directions: real analysis [15, 18], formal languages [13, 19], combinatorics on words [16, 20], recursive function theory [7, 8]. This paper adds to this heritage.

I was extremely privileged to have been part of the circle of people influenced by Professor Marcus during these more than 25 years. To him, all my gratitude and my best wishes.

Conclusions and Acknowledgment

Some of the concepts in this paper would deserve, we believe, further investigations, e.g. the (two versions of) $\mathbf{Q}$-differentiability. For instance, a natural question is whether the $\mathbf{Q}$-derivatives of a $\mathbf{Q}$-differentiable function satisfy some weak version of Darboux property.

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