Short Proofs of the Kneser-Lovász Coloring Principle

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Abstract. We prove that the propositional translations of the Kneser-Lovász theorem have polynomial size extended Frege proofs and quasi-polynomial size Frege proofs. We present a new counting-based combinatorial proof of the Kneser-Lovász theorem that avoids the topological arguments of prior proofs. We introduce a miniaturization of the octahedral Tucker lemma, called the *truncated Tucker lemma*. The propositional translations of the truncated Tucker lemma are polynomial size and they imply the Kneser-Lovász principles with polynomial size Frege proofs. It is open whether they have (quasi-)polynomial size Frege or extended Frege proofs.

1 Introduction

This paper discusses proofs of Lovász's theorem about the chromatic number of Kneser graphs, and the proof complexity of propositional translations of the Kneser-Lovász theorem. We give a new proof of the Kneser-Lovász theorem that uses a simple counting argument instead of the topological arguments used in prior proofs. Our arguments can be formalized in propositional logic to give polynomial size extended Frege proofs and quasi-polynomial size Frege proofs.

Frege systems are sound and complete proof systems for propositional logic with a finite schema of axioms and inference rules. The typical example is a "textbook style" propositional proof system using *modus ponens* as its only rule of inference, and all Frege systems are polynomially equivalent to this system. Extended Frege systems are Frege systems augmented with the extension rule,

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which allows variables to abbreviate complex formulas. Frege proofs are able to reason using Boolean formulas; whereas extended Frege proofs can reason using Boolean circuits. Since Boolean formulas are conjectured to require exponential size to simulate Boolean circuits, it is generally conjectured that there is an exponential separation between the sizes of Frege proofs and extended Frege proofs. This is one of the most important open questions in proof complexity.

However, as discussed by Bonet, Buss and Pitassi [2] and more recently by [1, 3], we do not have any examples of combinatorial tautologies that are conjectured to exponentially separate Frege proof size from extended Frege proof size. These prior works discussed the complexity of propositional proofs of a number of combinatorial principles, including the pigeonhole principle and Frankl's theorem. The present paper extends this work by giving short Frege proofs of the propositional translations of the Kneser-Lovász theorem. These upper bounds are of interest, in part because the give a novel proof of the Kneser-Lovász theorem, and in part because they help us understand the question of whether Frege systems (quasi-)polynomially simulate extended Frege systems.

Istrate and Crãciun [8] proposed the tautologies based on the Kneser-Lovász theorem as a candidate for a family of tautologies that could exponentially separate Frege and extended Frege proof sizes. These tautologies are of interest because the known proofs of the Kneser-Lovász theorem use (at least implicitly) a topological fixed-point lemma. The most combinatorial proof is by Matoušek [11] and is inspired by the octahedral Tucker lemma. Ziegler [13] gives other combinatorial proofs, but they also are inspired by topological constructions.

Our new proofs mostly avoid topological arguments and use a counting argument instead. The counting arguments reduce the general case of the Kneser-Lovász theorem to "small" instances of size $n \leq 2k^4$. For fixed k, there are only finitely many small instances, and they can be verified by exhaustive enumeration. For fixed values of k, these counting-based proofs can be converted into polynomial size extended Frege proofs and quasi-polynomial size Frege proofs. This shows that the Kneser-Lovász theorem does not give an exponential separation between Frege and extended Frege proof size.

It is surprising that the topological arguments can be largely eliminated from the proof of the Kneser-Lovász theorem. The only remaining use of topological arguments is to establish the "small instances". It would be interesting to give an additional argument that avoids having to prove the small instances separately.

Of independent interest, we also present a miniaturized version of the octahedral Tucker lemma called the *truncated Tucker lemma*. When the usual Tucker lemma is translated into propositional formulas, the resulting formulas are exponentially large. In contrast, the truncated Tucker lemma has polynomial size propositional translations. We prove that the Kneser-Lovász tautologies have polynomial size constant depth Frege proofs if the propositional formulas for truncated Tucker lemma are given as additional hypotheses. It remains open whether these truncated Tucker lemma principles have polynomial size Frege or extended Frege proofs.

The (n, k)-Kneser graph is defined to be the undirected graph whose vertices are the k-subsets of $\{1, \ldots, n\}$; there is an edge between two vertices iff those vertices have empty intersection. The Kneser-Lovász theorem states that Kneser graphs have a large chromatic number:

Theorem 1 (Lovász [10]). *Let $n \geq 2k > 1$. The (n, k)-Kneser graph has no coloring with $n - 2k + 1$ colors.*

It is easy to show that the (n, k)-Kneser graph has a coloring with $n - 2k + 2$ colors (see Sect. 6), so the bound $n - 2k + 1$ is optimal. For $k = 1$, the Kneser-Lovász theorem is just the pigeonhole principle.

Istrate and Crãciun [8] noted that, for fixed values of k, the propositional translations of the Kneser-Lovász theorem have polynomial size in n. They presented arguments that can be formalized by polynomial size Frege proofs for $k = 2$, and by polynomial size extended Frege proofs for $k = 3$. This left open the possibility that the $k = 3$ case could exponentially separate the Frege and extended Frege systems. It was also left open whether the $k > 3$ case of the Kneser-Lovász theorem gave tautologies that require exponential size extended Frege proofs. As discussed above, the present paper refutes these possibilities, as the next two theorems state the existence of polynomial size extended Frege proofs and quasi-polynomial size Frege proofs. Consequently, the Kneser-Lovász theorem is not a candidate for exponentially separating the Frege and extended Frege systems.

Theorem 2. *For fixed parameter $k \geq 1$, propositional translations of the Kneser-Lovász theorem have polynomial size extended Frege proofs.*

Theorem 3. *For fixed parameter $k \geq 1$, propositional translations of the Kneser-Lovász theorem have quasi-polynomial size Frege proofs.*

It would be interesting to know if there are short Frege proofs of the Kneser-Lovász theorem based on the argument in Matoušek [11]. The obstacle in doing so is that the propositional translations of the octahedral Tucker lemma are of exponential size in n. Sect. 4 introduces the truncated Tucker lemma, which has a polynomial size translation into propositional formulas. The truncated Tucker lemma is shown to follow from the Tucker lemma, and the Kneser-Lovász theorem follows from the truncated Tucker lemma. It is open whether the truncated Tucker lemma has short Frege or extended Frege proofs.

1.1 Preliminaries

Let $[n]$ be the set $\{1, \ldots, n\}$; members of $[n]$ are called "nodes". We identify $\binom{n}{k}$ with the set of k-subsets of $[n]$. The (n, k)-Kneser graph is the undirected graph with vertex set $\binom{n}{k}$ and with edges $\{S, T\}$ such that $S, T \in \binom{n}{k}$ and $S \cap T = \emptyset$.

Definition 4. *An* m-coloring *of the (n, k)-Kneser graph is a map c from $\binom{n}{k}$ to $[m]$, such that for $S, T \in \binom{n}{k}$, if $S \cap T = \emptyset$, then $c(S) \neq c(T)$. If $\ell \in [m]$ is a color, then the* color class P_ℓ *is the set $c^{-1}(\{\ell\})$.*

Definition 5. *A color class P_ℓ is* star-shaped *if $\bigcap P_\ell$ is non-empty. If P_ℓ is star-shaped, then any $i \in \bigcap P_\ell$ is called a* central element *of P_ℓ.*

The next lemma upper bounds the size of color classes that are not star-shaped. It will be used in our proof of the Kneser-Lovász theorem to establish the existence of star-shaped color classes. The idea is that non-star-shaped are too small to cover all $\binom{n}{k}$ tuples with only non-star-shaped color classes.

Lemma 6. *Let c be an m-coloring of $\binom{n}{k}$. If P_ℓ is not star-shaped, then*

$$|P_\ell| \;\leq\; k^2 \binom{n-2}{k-2}.$$

Proof. Suppose P_ℓ is not star-shaped. If P_ℓ is empty, the claim is trivial. So suppose $P_\ell \neq \emptyset$, and let $S_0 = \{a_1, \ldots, a_k\}$ be some element of P_ℓ. Since P_ℓ is not star-shaped, there must be sets $S_1, \ldots, S_k \in P_\ell$ with $a_i \notin S_i$ for $i = 1, \ldots, k$.

To specify an arbitrary element S of P_ℓ, we do the following. Since $S \cap S_0$ is non-empty, we first specify some $a_i \in S \cap S_0$. As $S \cap S_i$ is non-empty, we then specify some $a_j \in S \cap S_i$. By construction, $a_i \neq a_j$, so S is fully specified by the k possible values for a_i, the k possible values for a_j, and the $\binom{n-2}{k-2}$ possible values for the remaining members of S. Therefore, $|P_\ell| \leq k^2\binom{n-2}{k-2}$. □

Frege systems [7] are "textbook style" sound and implicationally complete propositional proof systems for tautologies. They have a finite schema of axioms and inference rules. The standard rule of inference is *modus ponens*: from A and $A \rightarrow B$, infer B. Extended Frege systems are Frege systems augmented with the extension rule, which allows a formula to be abbreviated by a new propositional variable. The size of a Frege or extended Frege proof is measured by counting the number of symbols in the proof [6]. We will not write out in detail the Frege and extended Frege proofs of Theorems 2 and 3. Instead we will give informal arguments that are straightforward to translate into Frege or extended Frege proofs using the techniques in [4]. More background on propositional proof systems is given in the surveys [5, 9, 12].

The formulas Kneser^n_k are the natural propositional translations of the statement that there is no $(n-2k+1)$-coloring of the (n,k)-Kneser graph:

Definition 7. *Let $n \geq 2k > 1$, and $m = n-2k+1$. For $A \in \binom{n}{k}$ and $i \in [m]$, the propositional variable $p_{A,i}$ has the intended meaning that vertex A of the Kneser graph is assigned the color i. The formula Kneser^n_k is*

$$\bigwedge_{A \in \binom{n}{k}} \bigvee_{i \in [m]} p_{A,i} \;\rightarrow\; \bigvee_{\substack{A,B \in \binom{n}{k} \\ A \cap B = \emptyset}} \bigvee_{i \in [m]} (p_{A,i} \wedge p_{B,i}).$$

2 Mathematical Arguments

This section gives our new proofs of the Kneser-Lovász theorem. Section 2.1 gives the first version, which we later show is formalizable as polynomial size extended Frege proofs. Section 2.2 gives a slightly more complicated but more efficient proof, which is later shown to be formalizable with quasi-polynomial size Frege proofs.

2.1 Argument for Extended Frege Proofs

This section gives our new proof of the Kneser-Lovász theorem. Sect. 3.1 will outline its formalization with polynomial size extended Frege proofs.

Let $k > 1$ be fixed. We prove the Kneser-Lovász theorem by induction on n. The base cases for the induction are $n = 2k, \dots, N(k)$ where $N(k)$ is the constant depending on k specified in Lemma 8. We shall show that $N(k)$ is no greater than k^4. Since k is fixed, there are only finitely many base cases. Since the Kneser-Lovász theorem is true, these base cases can all be proved by a fixed Frege proof of finite size (depending on k). Therefore, in our proof below, we only show the induction step.

Lemma 8. *Fix $k > 1$. There is an $N(k)$ so that, for $n > N(k)$, any $(n-2k+1)$-coloring of $\binom{n}{k}$ has at least one star-shaped color class.*

Proof. Suppose that a coloring c does not have any star-shaped color class. Since there are $n - 2k + 1$ many color classes, Lemma 6 implies that

$$(n - 2k + 1) \cdot k^2 \binom{n-2}{k-2} \;\geq\; \binom{n}{k}. \tag{1}$$

For fixed k, the left hand side of (1) is $\Theta(n^{k-1})$ and the right-hand side is $\Theta(n^k)$. Thus, there exists an $N(k)$ such that (1) fails for all $n > N(k)$. Hence for $n > N(k)$, there must be at least one star-shaped color class. □

We are now ready to give our first proof of the Kneser-Lovász theorem.

Proof (of Theorem 1). Fix $k > 1$. By Lemma 8, there is some $N(k)$ such that for $n > N(k)$, any $(n - 2k + 1)$-coloring c of $\binom{n}{k}$ has a star-shaped color. As discussed above, the cases of $n \leq N(k)$ cases are handled by exhaustive search and the truth of the Kneser-Lovász theorem. For $n > N(k)$, we prove the claim by infinite descent. In other words, we show that if c is an $(n - 2k + 1)$-coloring of $\binom{n}{k}$, then there is some c' which is an $((n - 1) - 2k + 1)$-coloring of $\binom{n-1}{k}$.

By Lemma 8, the coloring c has some star-shaped color class P_ℓ with central element i. Without loss of generality, $i = n$ and $\ell = n - 2k + 1$. Letting

$$c' \;=\; c \upharpoonright \tbinom{n-1}{k}$$

be the restriction of c to the domain $\binom{n-1}{k}$, it is clear that c' is an $((n-1)-2k+1)$-coloring of $\binom{n-1}{k}$. This completes the proof. □

For completeness, let us calculate an upper bound on the value of $N(k)$. Equation (1) is equivalent to

$$(n - 2k + 1)k^3(k - 1) \;\geq\; n(n - 1). \tag{2}$$

Since $2k - 1 \geq 1$, (2) implies that $(n - 1)k^4 > n(n - 1)$ and thus that $n < k^4$. Thus, (1) will be false if $n \geq k^4$; so $N(k) < k^4$.

2.2 Argument for Frege Proofs

We now give a second proof of the Kneser-Lovász theorem. The proof above required $n - N(k)$ rounds of infinite descent to transform a Kneser graph on n nodes to one on $N(k)$ nodes. Our second proof replaces this with only $O(\log n)$ many rounds, and this efficiency will be key for formalizing this proof with quasi-polynomial size Frege proofs in Sect. 3.2.

We refine Lemma 8 to show that for n sufficiently large, there are many (i.e., a constant fraction) shar-shaped color classes. The idea is to combine the upper bound of Lemma 6 on the size of non-star-shaped color classes with the trivial upper bound of $\binom{n-1}{k-1}$ on the size of star-shaped color classes.

Lemma 9. *Fix $k > 1$ and $0 < \beta < 1$. Then there exists an $N(k,\beta)$ such that for $n > N(k,\beta)$, if c is an $(n-2k+1)$-coloring of $\binom{n}{k}$, then c has at least $\frac{n}{k}\beta$ many star-shaped color classes.*

The value of $N(k,\beta)$ can be set equal to $\frac{k^3(k-\beta)}{1-\beta}$. The proof of Theorem 3 applies Lemma 9 with $\beta = 1/2$. Lemma 9 implies that for $n > N(k,1/2)$, any $(n-2k+1)$-coloring of $\binom{n}{k}$ has at least $\frac{n}{2k}$ many star-shaped color classes.

Proof (of Lemma 9). Let $n > N(k,\beta) = \frac{k^3(k-\beta)}{1-\beta}$, and suppose c is an $(n-2k+1)$-coloring of $\binom{n}{k}$. Let α be the number of star-shaped color classes of c. It is clear that an upper bound on the size of each star-shaped color class is $\binom{n-1}{k-1}$. There are $n-\alpha-2k+1$ many star-shaped classes, and Lemma 6 upper bounds their size by $k^2\binom{n-2}{k-2}$. This implies that

$$\binom{n-1}{k-1}\alpha + k^2\binom{n-2}{k-2}(n-\alpha-2k+1) \;\geq\; \binom{n}{k}. \tag{3}$$

Assume for a contradiction that $\alpha < \frac{n}{k}\beta$. Since $n > \frac{k^3(k-\beta)}{1-\beta}$, $0 < \beta < 1$, and $k \geq 2$, we have $n-1 > k^3(k-1) > k^2(k-1)$. Therefore, $\binom{n-1}{k-1} > k^2\binom{n-2}{k-2}$, and if α is replaced by the larger value $\frac{n}{k}\beta$, the left hand side of (3) increases. Thus,

$$\binom{n-1}{k-1}\frac{n}{k}\beta + k^2\binom{n-2}{k-2}\left(n - \frac{n}{k}\beta - 2k + 1\right) \;>\; \binom{n}{k}.$$

Since $\binom{n-1}{k-1}\frac{n}{k} = \binom{n}{k}$ and $n - \frac{n}{k}\beta - 2k + 1 = \frac{k-\beta}{k}n - 2k + 1$,

$$k^2\binom{n-2}{k-2}\left(\frac{k-\beta}{k}n - 2k + 1\right) \;>\; (1-\beta)\binom{n}{k}.$$

We have $\frac{k-\beta}{k}(n-1) > \frac{k-\beta}{k}n - 2k + 1$. Therefore,

$$k^3(k-1)\frac{k-\beta}{k}(n-1) \;>\; (1-\beta)n(n-1).$$

Dividing by $n-1$ gives $k^3(k-\beta) \;>\; (1-\beta)n$, contradicting $n > \frac{k^3(k-\beta)}{1-\beta}$. □

We now give our second proof of the Kneser-Lovász theorem.

Proof (of Theorem 1). Fix $k > 1$. By Lemma 9, if $n > N(k, 1/2)$ and c is an $(n - 2k + 1)$-coloring of $\binom{n}{k}$, then c has at least $n/2k$ many star-shaped color classes. We prove the Kneser-Lovász theorem by induction on n. The base cases are for $2k \le n \le N(k, 1/2)$, and there are only finitely of these, so they can be exhaustively proven. For $n > N(k, 1/2)$, we structure the induction proof as an infinite descent. In other words, we show that if c is an $(n - 2k + 1)$-coloring of $\binom{n}{k}$, then there is some c' that is an $((n - \frac{n}{2k}) - 2k + 1)$-coloring of $\binom{n-\frac{n}{2k}}{k}$. For simplicity of notation, we assume $\frac{n}{2k}$ is an integer. If this is not the case, we really mean to round up to the nearest integer $\lceil \frac{n}{2k} \rceil$.

By permuting the color classes and the nodes, we can assume w.l.o.g. that the $\frac{n}{2k}$ color classes P_ℓ for $\ell = n - \frac{n}{2k} - 2k + 2, \ldots, n - 2k + 1$ are star-shaped, and each such P_ℓ has central element $\ell + 2k - 1$. That is, the last $\frac{n}{2k}$ many color classes are star-shaped and their central elements are the last $\frac{n}{2k}$ nodes in $[n]$.

Define c' to be the coloring of $\binom{n-n/2k}{k}$ which assigns the same colors to k-tuples as c. The map c' is a $(\frac{2k-1}{2k}n - 2k + 1)$-coloring of $\binom{\frac{2k-1}{2k}n}{k}$, since $n - \frac{n}{2k} = \frac{2k-1}{2k}n$. This completes the proof of the induction step. □

When formalizing the above argument as quasi-polynomial size Frege proof, it will be important to know how many iterations of the procedure are required to reach the base cases, so let us calculate this.

After s iterations of this procedure, we have a $((\dfrac{2k-1}{2k})^s n - 2k + 1)$-coloring of $\binom{(\frac{2k-1}{2k})^s n}{k}$. We pick s large enough so that $(\frac{2k-1}{2k})^s n$ is less than $N(k, 1/2)$. In other words, since k is constant,

$$s \;=\; \log_{\frac{2k}{2k-1}} \frac{n}{k^3(2k-1)} \;=\; O(\log n)$$

will suffice, and only $O(\log n)$ many rounds of the procedure are required.

We do not know if the bound in Lemma 9 is optimal or close to optimal. Appendix 6 discusses the best examples we know of colorings with large numbers of non-star-shaped color classes.

3 Formalization in Propositional Logic

3.1 Polynomial Size Extended Frege Proofs

We sketch the formalization of the argument in Sect. 2.1 as a polynomial size extended Frege proof, establishing Theorem 2. We concentrate on showing how to express concepts such as "star-shaped color class" with polynomial size propositional formulas. We leave to the reader to check the details of how (extended) Frege proofs can prove properties of these concepts.

Fix values for k and n with $n > N(k)$. We describe an extended Frege proof of Kneser_k^n. We have variables $p_{S,j}$ (recall Definition 7), collectively denoted just $\vec{p}$. The proof assumes $\text{Kneser}_k^n(\vec{p})$ is false, and proceeds by contradiction. The main step is to define new variables $\vec{p}'$ and prove that $\text{Kneser}_k^{n-1}(\vec{p}')$ fails. This will be repeated until reaching a Kneser graph over only $N(k)$ nodes.

For this, let $\text{StarShaped}(i, \ell)$ be a formula that is true when $i \in [n]$ is a central element of the color class P_ℓ; namely,

$$\text{StarShaped}(i, \ell) \ := \bigwedge_{S \in \binom{n}{k},\, i \notin S} \neg p_{S,\ell}.$$

We use $\text{StarShaped}(\ell) := \bigvee_i \text{StarShaped}(i, \ell)$ to express that P_ℓ is star-shaped.

The extended Frege proof defines the instance of the Kneser-Lovasz principle Kneser_k^{n-1} by discarding one node and one color. The first star-shaped color class P_ℓ is discarded; accordingly, we let

$$\text{DiscardColor}(\ell) \ := \ \text{StarShaped}(\ell) \wedge \bigwedge_{\ell' < \ell} \neg \text{StarShaped}(\ell').$$

The node to be discarded is the least central element of the discarded P_ℓ:

$$\text{DiscardNode}(i) \ := \ \bigvee_{\ell} \Big[\text{DiscardColor}(\ell) \wedge \text{StarShaped}(i, \ell) \wedge \bigwedge_{i' < i} \neg \text{StarShaped}(i', \ell) \Big].$$

After discarding the node i and color class P_ℓ, the remaining nodes and colors are renumbered to the ranges $[n-1]$ and $[n-2k]$, respectively. In particular, the "new" color j (in the instance of Kneser_k^{n-1}) corresponds to the "old" color $j^{-\ell}$ (in the instance of Kneser_k^n) where

$$j^{-\ell} = \begin{cases} j & \text{if } j < \ell \\ j+1 & \text{if } j \geq \ell. \end{cases}$$

And, if $S = \{i_1, \ldots, i_k\} \in \binom{n-1}{k}$ is a "new" vertex (for the Kneser_k^{n-1} instance), then it corresponds to the "old" vertex $S^{-i} \in \binom{n}{k}$ (for the instance of Kneser_k^n), where $S^{-i} = \{i'_1, i'_2, \ldots, i'_k\}$ with

$$i'_t = \begin{cases} i_t & \text{if } i_t < i \\ i_t + 1 & \text{if } i_t \geq i. \end{cases}$$

For each $S \in \binom{n-1}{k}$ and $j \in [n-1]$, the extended Frege proof uses the extension rule to introduce a new variable $p'_{S,j}$ defined as follows

$$p'_{S,j} \equiv \bigvee_{i,\ell} \left(\text{DiscardNode}(i) \wedge \text{DiscardColor}(\ell) \wedge p_{S^{-i}, j^{-\ell}} \right).$$

As seen in the definition by extension, $p'_{S,j}$ is defined by cases, one for each pair i, ℓ of nodes and colors such that the node i is the least central element of the P_ℓ color class, where P_ℓ is the first star-shaped color class. The extended Frege proof then shows that $\neg\mathrm{Kneser}_k^n(\vec{p})$ implies $\neg\mathrm{Kneser}_k^{n-1}(\vec{p}')$, i.e., that if the variables $p_{S,j}$ define a coloring, then the variables $p'_{S,j}$ also define a coloring. For this, it is necessary to show that there is at least one star-shaped color class; this is provable with a polynomial size extended Frege proof (even a Frege proof) using the construction of Lemma 8 and the counting techniques of [4].

The extended Frege proof iterates this process of removing one node and one color until it is shown that there is a coloring of $\binom{N(k)}{k}$. This is then refuted by exhaustively considering all graphs with $\leq N(k)$ nodes. $\square$

3.2 Quasi-polynomial Size Frege Proofs

This section discusses some of the details of the formalization of the argument in Sect. 2.2 as quasi-polynomial size Frege proofs, , establishing Theorem 3. First we will form an extended Frege proof, then modify it to become a Frege proof. As before, the proof starts with the assumption that $\mathrm{Kneser}_k^n(\vec{p})$ is false. As we describe next, the extended Frege proof then introduces variables $\vec{p}'$ by extension so that $\mathrm{Kneser}_k^{n-n/2k}$ is false. This process will be repeated $O(\log n)$ times. The final Frege proof is obtained by unwinding the definitions by extension.

For a set of formulas X, and for $t > 0$, let "$|X| < t$" denote a formula that is true when the number of true formulas in X is less than t. If there are polynomially many formulas in X, each of polynomial size, then $|X| < t$ is of polynomial size using the construction in [4]. The formula "$|X| = t$" is defined similarly.

The formulas $\mathrm{StarShaped}(i,\ell)$ and $\mathrm{StarShaped}(\ell)$ are the same as in Sect. 3.1. A color ℓ is now discarded if it is among the least $n/2k$ star-shaped color classes.

$$\mathrm{DiscardColor}(\ell) \;:=\; \mathrm{StarShaped}(\ell) \wedge (|\{\mathrm{StarShaped}(\ell') :\; \ell' \leq \ell\}| \leq n/2k)$$

The discarded nodes are the least central elements of the discarded color classes.

$$\mathrm{DiscardNode}(i) \;:=\; \bigvee_{\ell} \Big[\mathrm{DiscardColor}(\ell) \wedge \mathrm{StarShaped}(i,\ell) \wedge \bigwedge_{i'<i} \neg\mathrm{StarShaped}(i',\ell) \Big].$$

The remaining, non-discarded colors and nodes are renumbered to form an instance of $\mathrm{Kneser}_k^{n-n/2k}$. For this, the formula $\mathrm{RenumNode}(i',i)$ is true when the node i' is the ith node that is not discarded; similarly $\mathrm{RenumColor}(j',j)$ is true when the color j' is the jth color that is not discarded.

$$\mathrm{RenumNode}(i',i) := (|\{\neg\mathrm{DiscardNode}(i'') : i'' < i'\}| = i-1) \wedge \neg\mathrm{DiscardNode}(i')$$

$$\mathrm{RenumColor}(j',j) := (|\{\neg\mathrm{DiscardColor}(j'') : j'' < j'\}| = j-1) \wedge \neg\mathrm{DiscardColor}(j')$$

For each $S = \{i_1, \ldots, i_k\} \in \binom{n-n/2k}{k}$ and $j \in [(n-n/2k)-2k+1]$, we define by extension

$$p'_{S,j} \equiv \bigvee_{i'_1,\ldots i'_k, j'} \left(\bigwedge_{t=1}^{k} (\mathrm{RenumNode}(i'_t, i_t)) \wedge \mathrm{RenumColor}(j',j) \wedge p_{\{i'_1,\ldots,i'_k\},j'} \right).$$

The Frege proof then argues that if the variables $p_{S,j}$ define a coloring, then the variables $p'_{S,j}$ define a coloring, i.e., that $\neg\text{Kneser}^n_k(\vec{p}) \to \neg\text{Kneser}^{n-n/2k}_k(\vec{p}')$. The main step for this is proving there are at least $n/2k$ star-shaped color classes by formalizing the proof of Lemma 9; this can be done with polynomial size Frege proofs using the counting techniques from [4]. After that, it is straightforward to prove that, for each $S \in \binom{n-n/2k}{k}$ and $j \in [(n-n/2k)-2k+1]$, the variable $p'_{S,j}$ is well-defined; and that the $\vec{p}'$ collectively falsify $\text{Kneser}^{n-n/2k}_k$.

This is iterated $O(\log n)$ times until fewer than $N(k,1/2)$ nodes remain. The proof concludes with a hard-coded proof that there are no such colorings of small Kneser graphs.

To form the quasi-polynomial size Frege proof, we unwind the definitions by extension. Each definition by extension was polynomial size; they are nested to a depth of $O(\log n)$. So the resulting Frege proof is quasi-polynomial size. □

4 The Truncated Tucker Lemma

This section introduces the truncated Tucker lemma. We show that the usual Tucker lemma implies the truncated Tucker lemma and the truncated Tucker lemma implies the Kneser-Lovász theorem. The truncated Tucker lemma is of particular interest, since its propositional translations are only polynomial size; in contrast, the propositional translations of the usual Tucker lemma are of exponential size. Additionally, there are polynomial size constant depth Frege proofs of the Kneser-Lovász tautologies from the truncated Tucker tautologies.

Our proof of the truncated Tucker lemma from the Tucker lemma borrows techniques and notation from Matoušek [11].

Definition 10. *Let $n \geq 1$. The* octahedral ball $\mathcal{B}^n$ *is:*

$$\mathcal{B}^n \ := \ \{(A,B) : A, B \subseteq [n] \text{ and } A \cap B = \emptyset\}.$$

Let $1 \leq k \leq n$. The truncated octahedral ball $\mathcal{B}^n_k$ *is:*

$$\mathcal{B}^n_k \ := \ \Big\{(A,B) : A, B \in \binom{n}{k} \cup \{\emptyset\},\ A \cap B = \emptyset,\ \text{and } (A,B) \neq (\emptyset,\emptyset)\Big\}.$$

Definition 11. *Let $n > 1$. A mapping $\lambda : \mathcal{B}^n \to \{1, \pm 2, \ldots, \pm n\}$ is* antipodal *if $\lambda(\emptyset,\emptyset) = 1$, and for all other pairs $(A,B) \in \mathcal{B}^n$, $\lambda(A,B) = -\lambda(B,A)$.*

Let $n \geq 2k > 1$. A mapping $\lambda : \mathcal{B}^n_k \to \{\pm 2k, \ldots, \pm n\}$ is antipodal *if for all $(A,B) \in \mathcal{B}^n_k$, $\lambda(A,B) = -\lambda(B,A)$.*

Note that -1 is not in the range of λ, and $(\emptyset,\emptyset)$ is the only member of $\mathcal{B}^n$ that is mapped to 1 by λ.

For $A \subseteq [n]$, let $A_{\leq k}$ denote the least k elements of A. By convention $\emptyset_{\leq k} = \emptyset$, but otherwise the notation is used only when $|A| \geq k$.

Definition 12. *Let $\preceq$ be the partial order on sets in $\binom{n}{k} \cup \{\emptyset\}$ defined by $A_1 \preceq A_2$ iff $(A_1 \cup A_2)_{\leq k} = A_2$.*

As we will see momentarily in Definition 14 and Lemmas 15 and 16, the Tucker lemma uses the subset relation $\subseteq$ on $[n]$, but the truncated Tucker lemma uses instead the stronger partial order $\preceq$ on $\binom{n}{k}$.

Lemma 13. *The relation $\preceq$ is a partial order with $\emptyset$ its least element.*

Proof. It is clearly reflexive. For anti-symmetry, $A_1 \preceq A_2$ and $A_2 \preceq A_1$ imply that $A_1 = (A_1 \cup A_2)_{\le k} = (A_2 \cup A_1)_{\le k} = A_2$. For transitivity: Suppose $A_1 \preceq A_2$ and $A_2 \preceq A_3$. Then $(A_1 \cup A_2)_{\le k} = A_2$ and $(A_2 \cup A_3)_{\le k} = A_3$. This implies that

$$A_3 = (A_2 \cup A_3)_{\le k} = ((A_1 \cup A_2)_{\le k} \cup A_3)_{\le k} = (A_1 \cup (A_2 \cup A_3)_{\le k})_{\le k} = (A_1 \cup A_3)_{\le k}.$$

Therefore $A_1 \preceq A_3$. That $\emptyset$ is the least element is clear from the definition. □

Definition 14. *Two pairs (A_1, B_1) and (A_2, B_2) in $\mathcal{B}^n$ are* complementary *w.r.t. an antipodal map λ on $\mathcal{B}^n$ if $A_1 \subseteq A_2$, $B_1 \subseteq B_2$ and $\lambda(A_1, B_1) = -\lambda(A_2, B_2)$.*

For (A_1, B_1) and (A_2, B_2) in $\mathcal{B}^n_k$, write $(A_1, B_1) \preceq (A_2, B_2)$ when $A_1 \preceq A_2$, $B_1 \preceq B_2$, and $A_i \cap B_j = \emptyset$ for $i, j \in \{1, 2\}$. And, (A_1, B_1) and (A_2, B_2) are k-complementary *w.r.t. an antipodal map λ on $\mathcal{B}^n_k$ if $\lambda(A_1, B_1) = -\lambda(A_2, B_2)$.*

Lemma 15 (Tucker lemma). *If $\lambda : \mathcal{B}^n \to \{1, \pm 2, \ldots, \pm n\}$ is antipodal, then there exists two elements in $\mathcal{B}^n$ that are complementary.*

Lemma 16 (Truncated Tucker). *Let $n \ge 2k > 1$. If $\lambda : \mathcal{B}^n_k \to \{\pm 2k \ldots, \pm n\}$ is antipodal, then there are two elements in $\mathcal{B}^n_k$ that are k-complementary.*

For a proof of Lemma 15, see [11]. Appendix 5 proves Lemma 15 implies Lemma 16, hence the truncated Tucker lemma is true.

The truncated Tucker lemma has polynomial size propositional translations. For each $(A, B) \in \mathcal{B}^n_k$, and for each $i \in \{\pm 2k, \ldots, \pm n\}$, let $p_{A,B,i}$ be a propositional variable with the intended meaning that $p_{A,B,i}$ is true when $\lambda(A, B) = i$. The following formula $\mathrm{Ant}(\vec{p})$ states that the map is total and antipodal:

$$\bigwedge_{(A,B)\in\mathcal{B}^n_k} \ \bigvee_{i\in\{\pm 2k,\ldots,\pm n\}} (p_{A,B,i} \wedge p_{B,A,-i}).$$

The following formula $\mathrm{Comp}(\vec{p})$ states that there exists two elements in $\mathcal{B}^n_k$ that are k-complementary:

$$\bigvee_{\substack{(A_1,B_1),(A_2,B_2)\in\mathcal{B}^n_k,\\ (A_1,B_1)\preceq(A_2,B_2)\\ i\in\{\pm 2k,\ldots,\pm n\}}} (p_{A_1,B_1,i} \wedge p_{A_2,B_2,-i}).$$

The truncated Tucker tautologies are defined to be $\mathrm{Ant}(\vec{p}) \to \mathrm{Comp}(\vec{p})$. (We could add an additional hypothesis, that for each A, B there is at most one i such that $p_{A,B,i}$, but this is not needed for the Tucker tautologies to be valid.) There are $< n^{2k}$ members (A, B) in $\mathcal{B}^n_k$. Hence, for fixed k, there are only polynomially many variables $p_{A,B,i}$, and the truncated Tucker tautologies have size polynomially bounded by n. On the other hand, the propositional translation of the usual Tucker lemma requires an exponential number of propositional variables in n, since the cardinality of $\mathcal{B}^n$ is exponential in n.

Proof (Theorem 1 from the truncated Tucker lemma). Let $c : \binom{n}{k} \to \{2k, \ldots, n\}$ be a coloring of $\binom{n}{k}$. We show that this implies the existence of an antipodal map λ on $\mathcal{B}_k^n$ that has no k-complementary pairs. Let $\leq$ be a total order on $\binom{n}{k} \cup \{\emptyset\}$ that refines the partial order $\preceq$. Define $\lambda(A, B)$ to be $c(A)$ if $A > B$ and $-c(B)$ if $B > A$. We argue that there are no k-complementary pairs in $\mathcal{B}_k^n$ with respect to λ. Suppose there are, say (A_1, B_1) and (A_2, B_2). Since λ must assign these opposite signs, either $A_1 < B_1 \leq B_2 < A_2$ or $B_1 < A_1 \leq A_2 < B_2$. In the former case it must be that, $c(B_1) = c(A_2)$ and in the latter case that $c(A_1) = c(B_2)$. Since $B_1 \cap A_2$ and $A_1 \cap B_2$ are empty in either case we have a contradiction, since c was assumed to be a coloring. □

The above proof of the Kneser-Lovász theorem from the truncated Tucker lemma can be readily translated into polynomial size constant depth Frege proofs.

Question 17. Do the propositional translations of the Truncated Tucker lemma have short (extended) Frege proofs?

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5 Appendix: Proof of the Truncated Tucker Lemma

We prove the truncated Tucker lemma from the Tucker lemma:

Proof (of Lemma 16 from Lemma 15). We show the contrapositive. Suppose Lemma 16 is false. In other words, there is some antipodal $\lambda : \mathcal{B}_k^n \to \{\pm 2k, \ldots, \pm n\}$ and there are no k-complementary pairs of elements in $\mathcal{B}_k^n$ with respect to λ. We will define an antipodal map $\lambda' : \mathcal{B}^n \to \{1, \pm 2, \ldots, \pm n\}$ that has no complementary pairs of elements in $\mathcal{B}^n$ with respect to λ', in violation of Lemma 15.

Let $\leq$ be a total order on the subsets of $[n]$ that respects cardinalities and refines $\preceq$ on elements in $\binom{n}{k} \cup \{\emptyset\}$. In other words, if $|A| < |B|$ or if $A \preceq B$ then $A \leq B$. Similarly to a construction in [11], $\lambda'(A, B)$ is defined by cases:

Case I: If $|A| < k$ and $|B| < k$, then

$$\lambda'(A, B) = \begin{cases} 1 + |A| + |B| & \text{if } A \geq B \\ -(1 + |A| + |B|) & \text{if } B > A. \end{cases}$$

Case II: If $|A| = k$ and $|B| < k$, then $\lambda'(A, B) = \lambda(A, \emptyset)$. Similarly, if $|A| < k$ and $|B| = k$, then $\lambda'(A, B) = \lambda(\emptyset, B)$.

Case III: If $|A| \geq k$ and $|B| \geq k$, then $\lambda'(A, B) = \lambda(A_{\leq k}, B_{\leq k})$.

The map λ' is clearly antipodal since λ is. Let (A_1, B_1) and (A_2, B_2) be members of $\mathcal{B}^n$ with $A_1 \subseteq A_2$, $B_1 \subseteq B_2$. We wish to show they are not a complementary pair for λ', namely that $\lambda'(A_1, B_1) \neq -\lambda'(A_2, B_2)$. The argument splits into cases.

1. $\lambda'(A_1, B_1)$ and $\lambda'(A_2, B_2)$ are assigned by Case I. If $A_1 = A_2$ and $B_1 = B_2$, then clearly $\lambda'(A_1, B_1) \neq -\lambda'(A_2, B_2)$. Otherwise, at least one of the inclusions $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$ is proper, so $|A_1| + |B_1| < |A_2| + |B_2|$. But $|\lambda'(A_1, B_1)| = 1 + |A_1| + |B_1|$ and $|\lambda'(A_2, B_2)| = 1 + |A_2| + |B_2|$, so $\lambda'(A_1, B_1) \neq -\lambda'(A_2, B_2)$.
2. $\lambda'(A_1, B_1)$ and $\lambda'(A_2, B_2)$ are assigned by Case II. Without loss of generality, $|A_1| = k$ and $|B_1| < k$. So then $A_2 = A_1$ and $|B_2| < k$. But then $\lambda'(A_1, B_1) = \lambda(A_1, \emptyset)$ and $\lambda'(A_2, B_2) = \lambda(A_1, \emptyset)$, so $\lambda'(A_1, B_1) \neq -\lambda'(A_2, B_2)$.
3. $\lambda'(A_1, B_1)$ and $\lambda'(A_2, B_2)$ are both assigned by Case III. It is clear that $A_{i,\leq k} \cap B_{j,\leq k} = \emptyset$ for $i, j \in \{1, 2\}$, since $A_2 \cap B_2 = \emptyset$. We claim that $A_{1,\leq k} \preceq A_{2,\leq k}$ and $B_{1,\leq k} \preceq B_{2,\leq k}$. This is because $A_1 \subseteq A_2$, so

   $$(A_{1,\leq k} \cup A_{2,\leq k})_{\leq k} = (A_1 \cup A_2)_{\leq k} = A_{2,\leq k}.$$

   Therefore $A_{1,\leq k} \preceq A_{2,\leq k}$. The same argument shows that $B_{1,\leq k} \preceq B_{2,\leq k}$. Since there are no k-complementary pairs with respect to λ, it must be that $\lambda(A_{1,\leq k}, B_{1,\leq k}) \neq -\lambda(A_{2,\leq k}, B_{2,\leq k})$. Also, $\lambda'(A_1, B_1) = \lambda(A_{1,\leq k}, B_{1,\leq k})$ and $\lambda'(A_2, B_2) = \lambda(A_{2,\leq k}, B_{2,\leq k})$. Hence $\lambda'(A_1, B_1) \neq -\lambda'(A_2, B_2)$.

4. $\lambda'(A_1, B_1)$, and $\lambda'(A_2, B_2)$ are assigned by Cases II and III. If this happens, it must be that (A_1, B_1) is assigned as in Case II, and (A_2, B_2) is assigned as in Case III. Without loss of generality, say $|A_1| = k$, and $|B_1| < k$. We show that $(A_1, \emptyset) \preceq (A_2, B_{2,\leq k})$. By the same argument as the previous case, $A_{1,\leq k} \preceq A_{2,\leq k}$. As remarked earlier, $\emptyset \preceq B_{2,\leq k}$, since the empty set is the least element in the $\preceq$ partial order. Also, the four sets $A_{1,\leq k} \cap \emptyset$, $A_{1,\leq k} \cap B_{2,\leq k}$, $\emptyset \cap B_{2,\leq k}$, and $A_{2,\leq k} \cap B_{2,\leq k}$ are empty.
   Since there are no k-complementary pairs with respect to λ, it must be that $\lambda(A_{1,\leq k}, \emptyset) \neq -\lambda(A_{2,\leq k}, B_{2,\leq k})$. By definition, $\lambda'(A_1, B_1) = \lambda(A_{1,\leq k}, \emptyset)$ and $\lambda'(A_2, B_2) = \lambda(A_{2,\leq k}, B_{2,\leq k})$. Hence $\lambda'(A_1, B_1) \neq -\lambda'(A_2, B_2)$.
5. The only remaining case is when one of $\lambda'(A_1, B_1)$ and $\lambda'(A_2, B_2)$ is assigned by Case I, and the other is assigned by Case II or III. It must be that $\lambda'(A_1, B_1)$ is assigned by Case I and $\lambda'(A_2, B_2)$ by Case II or III. Observe that $|\lambda'(A_1, B_1)| = 1 + |A_1| + |B_1| \leq 2k - 1$, and that $|\lambda'(A_2, B_2)| \geq 2k$. Therefore $\lambda'(A_1, B_1) \neq -\lambda'(A_2, B_2)$.

This establishes that λ' is an antipodal map with no complementary pairs, hence Lemma 15 is false. This completes the proof of the contrapositive. □

6 Appendix: Optimal Colorings of Kneser Graphs

It is well-known that $\binom{n}{k}$ has an $(n-2k+2)$-coloring [10]. A simple construction of such a coloring, which we call c_1, is as follows. For $S \in \binom{n}{k}$, define $c_1(S)$ by:

(1) If $S \not\subseteq [2k-1]$, let $c_1(S) = \max(S) - (2k-2)$. Clearly $1 < c_1(S) \leq n-2k+2$.
(2) Otherwise, let $c_1(S) = 1$.

We claim that c_1 defines a proper coloring. By construction, if $c_1(S) > 1$, then $c_1(S) + (2k-2) \in S$. Thus, if $c_1(S) = c_1(S') > 1$, then $S \cap S' \neq \emptyset$ and S and S' are not joined by an edge in the Kneser graph. On the other hand, if $c_1(S) = 1$, then S contains k elements from the set $[2k-1]$. Any two such subsets have nonempty intersection, and therefore if $c_1(S) = c_1(S') = 1$, then again $S \cap S' \neq \emptyset$. Note that c_1 contains $n-2k+1$ many star-shaped color classes, and only one non-star-shaped color class.

In view of Lemma 9, it is interesting to ask whether it is possible to give $(n-2k+2)$-colorings with fewer star-shaped color classes and more non-star-shaped color classes. The next theorem gives the best construction we know.

Theorem 18. *Let $k \geq 1$ and $n \geq 3k+3$. There is an $(n-2k+2)$ coloring c_{k-1} of $\binom{n}{k}$ which has $k-1$ many non-star-shaped color classes and only $n-3k+3$ many star-shaped color classes.*

Proof. To construct c_{k-1}, partition the set $[n]$ into $n-2k+2$ many subsets $T_1, \ldots, T_{n-2k+2}$ as follows. For $i \leq n-3k+3$, T_i is chosen to be a singleton set, say $T_i = \{n-i+1\}$. The remaining $k-1$ many T_i's are subsets of size 3, say $T_i = \{j-2, j-1, j\}$ where $j = i-(n-3k+3)$. Since $n = (n-3k+3)+3(k-1)$, the sets T_i partition $[n]$, and each T_i has cardinality either 1 or 3. For S a subset of n of cardinality k, define the color $c_{k-1}(S)$ to equal the least i such that

$$|S \cap T_i| \;>\; \frac{1}{2}|T_i|.$$

We claim there must exist such an i. If not, then S contains no members of the singleton subsets T_i and at most one member of each of the subsets T_i of size three. But there are only $k-1$ many subsets of size three, contradicting $|S| = k$.

It is easy to check that if $c_{k-1}(S) = c_{k-1}(S')$ then $S \cap S' \neq \emptyset$. Thus c_{k-1} is a coloring. Furthermore, c_{k-1} has $k-1$ many non-star-shaped classes and $n-3k+3$ many star-shaped classes. □

Theorem 18 can be extended to show that when $n < 3k+3$, there is a $n-2k+2$ coloring with no star-shaped color class. The proof construction uses a similar idea, based on the fact that $[n]$ can be partitioned into $n-2k+2$ many subsets, each of odd cardinality ≥ 3. We leave the details to the reader.

Question 19. Do there exist $(n-2k+2)$-colorings of the (n,k)-Kneser graphs with more than $k-1$ many non-star-shaped color classes?