Interactive Particle Systems and Random Walks on Hypergraphs

Gabriel Istrate, Cosmin Bonchiş, Mircea Marin *

May 18, 2014

Abstract

We study hypergraph analogues of interacting particle systems and random walks, notably generalizations of coalescing and annihilating random walks. Their definition is motivated by the problem of analyzing the expected running time of a local search procedure for the k-XOR SAT problem, as well as a certain constrained triad dynamics in the theory of social balance.

1 Introduction

Interacting particle systems are discrete dynamical systems, usually defined on lattices, studied intensely in Mathematical Physics [Lig04]. They can be investigated on finite graphs as well [DW83, DW84] as finite Markov chains, and correspond via *duality* to certain types of random walks [AF14]. The analysis of these models can sometimes be used to bound the mixing time of certain (hyper)graph coloring procedures [DW84, CT13].

A recent development in interacting particle systems and random walks is the extension of the theory to hypergraphs [CT13, LP12, CD12, CFR13, ALL14] and simplicial complexes [SKM12, PRT12]. We contribute to this direction by studying hypergraph analogues of coalescing/annihilating random walks and the voter model.

*West University of Timişoara and the eAustria Research Institute, Bd. V. Pârvan 4, cam 045B, Timişoara, RO-300223, Romania. email:{gistrate,cbonchis,mmarin}@info.uvt.ro

Besides the obvious fundamental interest of such a generalization, the models we consider are motivated by several apparently unrelated applications: the analysis of a local search procedure for the XOR-SAT problem, the theory of social balance [AKR06] and that of lights-out games[Sch14]. On the other hand the study of these systems, though it preserves some properties from the graph case has additional interesting features: for instance for so-called annihilating random walks on hypergraphs the number of particles is **not** in general nondecreasing (as it is in the graph case) and the structure of recurrent states is interestingly constrained by systems of linear equations similar to the ones used to analyze lights-out games [Sch14]. On the other hand, in coalescing random walks on hypergraphs there may be more than one copy of an initial "ball" and the process is naturally described using *multisets* rather than sets of balls.

The plan of the paper is as follows: first we define the models we are interested in and outline their motivation. In Section 3 we present the (still open in general) issue of reachability and recurrence for annihilating random walks, together with a result settling this for our intended applications. In such a setting, our main result (Theorem 4 in Section 4) upper bounds expected annihilation time in terms of a Cheeger-like constant of the hypergraph. We conclude with an application of this result to the analysis of the running time of a RandomWalk algorithm for instances of k-XOR-SAT and other (brief) remarks.

2 Preliminaries and motivating examples

Hypergraphs considered in this paper are *simple*: for every two hyperedges e, f, $|e \cap f| \leq 1$. On the other hand we will allow *self-loops*, i.e. hyperedges e with $|e| = 1$. We will even allow multiple self-loops to the same vertex. A *multiset* is a set whose elements have a (positive) multiplicity. The *disjoint union* of multisets A and B, denoted $A \sqcup B$, is the multiset that adds up multiplicities of an element in A and B.

Definition 1. *Given constant $k \geq 2$, an instance of* k-XOR-SAT *is a linear system of boolean equations $A \cdot \overrightarrow{x} = \overrightarrow{b}$, where A is an $m \times n$ matrix, for some $m, n \geq 1$, $\overrightarrow{x} = (x_1, x_2, \ldots, x_n)^T$ is an $n \times 1$ vector, $\overrightarrow{b} = (b_1, b_2, \ldots, b_m)^T$ is an $m \times 1$ vector, and each equation has exactly k variables.*

Though k-XOR-SAT can be solved in polynomial time by Gaussian elim-

Algorithm RandomWalk(Φ):

Start with an arbitrary assignment U.
while (there exists some unsatisfied clause)
  pick a random unsatisfied clause C
  change the value of a random variable of C in U
return assignment U.

Figure 1: The RandomWalk algorithm

ination, we will analyze instead a local search procedure, the RandomWalk algorithm displayed in Figure 1. The analysis of local procedures is quite complicated in general, so performing such an analysis is, we feel, interesting. Indeed, we will obtain rigorous upper bounds on the expected running time of RandomWalk on solvable instances in terms of measurable parameters of these instances.

A second motivation comes from the physics of complex systems and is given by the following dynamics:

Definition 2. Constrained Triadic Dynamics*[AKR06, Ist09]. We start with a graph $G = (V, E)$ whose edges are labeled* $0/1$*. A triangle T is G is called* balanced *if the sum of the labels of its edges is 0 (modulo 2). At any step t, we chose an imbalanced triangle T uniformly at random and we change the sign of a random edge of T (thus making T balanced). The move might, however, make other triangles unbalanced.*

CTD can be modeled by the RandomWalk algorithm on an instance of 3-XOR-SAT [RVYMO06]. As further shown in [Ist09], one can sometimes analyze CTD using duality. We give here a slightly more general version, suitable for the analysis of k-XOR-SAT:

Definition 3. *Given instance Φ of k-XOR-SAT, the* dual $D(\Phi)$ *of Φ is an undirected hypergraph with self-loops $D(\Phi) = (\overline{V}, \overline{E})$ defined as follows: $\overline{V}$ is the set of equations of Φ. Hyperedges in $D(\Phi)$ correspond to variables in Φ and connect all equations containing a given variable. In particular we add a self-loop to an equation (vertex) v if it contains a variable appearing* only in v*. We may even add multiple self-loops to the same vertex.*

Note that if Φ is an instance of $k - XOR - SAT$ then $D(\Phi)$ is a k-**regular hypergraph.** When viewed by duality the RandomWalk algorithm translates to:

Definition 4 (Annihilating random walk (a.r.w.) on hypergraphs)**.** *Let $H = (V, E)$ be a connected hypergraph. Define a* annihilating random walk on H *by the following:* (a). Initial state: *Initially: $A_i \in \{0, 1\}$. We will call a vertex i with $A_i = 1$* live*.* (b). Moves: *Choose random node i and random edge $(i, j_1, \ldots, j_k)$ containing i. If node i is live then for $r = 1, \ldots, k$ make $A_j = 0$ if j is live, $A_j = 1$ otherwise. Also make $A_i = 0$. If node i is not live do not do anything.*

Moves of the RandomWalk algorithm on an instance of k-XORSAT correspond to moves of the a.r.w. from live nodes only. The upper bounds we provide will, of course, work for this quantity as well.

It will be useful to define an analogue of coalescing random walks to hypergraphs as well:

Definition 5 (Coalescing random walks (c.r.w.) on hypergraphs)**.** *Let $H = (V, E)$ be a connected hypergraph. Each vertex holds a multiset of label A_i. Define a* coalescing random walk on H *by the following:* (a). Initial state: *$A_i = \{i\}$. Note that $A_1 \cup A_2 \cup \ldots \cup A_n = [n]$. We will call a vertex i with $|A_i| = odd$* live*.* (b). Moves: *Choose random node i. Choose random hyperedge $e = (i, j_1, j_2, \ldots, j_k)$. Make $A_{j_r} := A_{j_r} \sqcup A_i$, for $r = 1, \ldots, k$, $A_i = \emptyset$. Here $\sqcup$ refers to the* **multiset union***, i.e. union with multiplicities. Note that the move never destroys any label (always $A_1 \cup A_2 \cup \ldots \cup A_n = [n]$) but may make some indices i satisfy $|A_i| = even$.* (c). Parity (coalescence): *$c_{coal}(H)$ is the minimum $t \geq 0$ such that $|A_j| = even$ for every j.*

Finally, we will need the "dual" to coalescing random walks:

Definition 6 (Voter model on hypergraphs)**.** *Let $H = (V, E)$ be a connected hypergraph. Define a* voter model on H *by the following:* (a). Initial state: *$A_i = \{i\}$. Note that $A_1 \cup A_2 \cup \ldots \cup A_n = [n]$. W* (b). Moves: *Choose random node i. Choose random hyperedge $e = (i, j_1, j_2, \ldots, j_k)$. Make $A_i = \sqcup_{r=1}^{k} A_{j_r}$. Note that the operation may decrease the number of different "opinions" present in the system, if such opinions were only held by node i.* (c). Parity of opinions: Parity time *$c_{VM}(H)$ is the minimum t such that every initial opinion appears an even number of times (perhaps zero times) in the system.*

3 Annihilating random walks: reachability and recurrence

If the hypergraph H is a graph the long-term structure of configurations of the a.r.w. is simple: either a single live site survives (if $|V(H)|$ is odd) or none. In the general case the behavior is more complicated: the number of live nodes is **not** necessarily decreasing, as is the case in the graph setting. There may be, therefore, recurrent states different from $\mathbf{0}$ and those states with a single live node.

The structure of recurrent states is easy to determine, though, for satisfiable instances of k-XOR-SAT:

Theorem 1. *Let Φ be a satisfiable instance of k-XOR-SAT. Let X_1 be an arbitrary assignment and let w_1 be the configuration in the hypergraph $D(\Phi)$ corresponding to X_1. Finally let w_2 be the "all-zeros" configuration. Then w_2 is reachable from w_1. In other words a satisfying assignment X_2 for Φ can be found from initial assignment X_1 by means of moves of the RandomWalk algorithm.*

In the general case setting can give [Ist09] a necessary condition for reachability:

Definition 7. *For every pair of boolean configurations $w_1, w_2 : V(H) \to \mathbf{Z}_2$ on hypergraph H we define a system of boolean linear equations $H(w_1, w_2)$ as follows: Define, for each hyperedge e a variable z_e with values in $\mathbf{Z}_2$. For any vertex $v \in V(H)$ we define the equation $\sum_{v\in e} z_e = w_2(v) - w_1(v)$. In the previous equation the difference on the right-hand side is taken in $\mathbf{Z}_2$; also, we allow empty sums on the left side. System $H(w_1, w_2)$ simply consists of all equations, for all $v \in V(H)$.*

Definition 8. *If x is a state on H and l is a hyperedge of H, define $x^{(l)}(v) = 1 + x(v)$, if $v \in l$, $x(v)$, otherwise.*

Lemma 1. *If state w_2 is reachable from w_1 then the system of equations $H(w_1, w_2)$ has a solution in $\mathbf{Z}_2$.*

Proof. Let P be a path from w_1 to w_2 and let z_e be the number of times edge e is used on path P (mod 2). Then $(z_e)_{e\in E}$ is a solution of system $H(w_1, w_2)$. Indeed, element $w(v)$ (viewed modulo 2) flips its value anytime an edge containing v is scheduled.

□

In [Ist09] we claimed a partial converse of Lemma 1. As the result below shows, though, the converse of Lemma 1 is however **not** true in graphs, or even in hypergraphs with no graph edges:

Lemma 2. *The following are true: There exists (a). a connected graph (i.e. all hyperedges have size two) H, or (b). a connected hypergraph H that contains no graph edges; there also exist two configurations w_1, w_2 on H such that system $H(w_1, w_2)$ has solutions in $\mathbf{Z}_2$, yet w_2 is not reachable in H from w_1.*

Proof. Assume that X is a solution to the system $A \cdot x = b$. Let X_0 be an initial assignment. We will prove that a solution of the system is reachable from X_0 by induction on k, the Hamming distance between X_0 and X.

- **Case** $k = 0$**.** Then $X_0 = X$ and there is nothing to prove.
- **Case** $k = 1$**.** Then X_0 and X differ on a single variable z. Let w be an equation containing z. Then X_0 does not satisfy w (as X, which only differs on z, does). Choosing equation w and variable z we reach X from X_0.
- **Case** $k \geq 2$. If there is an equation w not satisfied by X_0 (but satisfied by X) then w must contain a variable on which X_0 and X differ. Let z be such a variable. Then by flipping z one can reach from X_0 an assignment X_1 at Hamming distance $k - 1$ from X. Now it is easily seen that system $H(X_1, X)$ has solutions: any solution of $H(X_0, X)$ with the value of z flipped. By the induction hypothesis one can reach a solution from X_1, therefore from X_0.

□

While we raise the complexity of reachability as an open problem, we believe it is possible to "patch" the result in [Ist09] (perhaps by imposing meaningful restrictions on states w_1, w_2) and further extend it in order to provide a large class of reachability instances for which the necessary condition in Lemma 1 is also sufficient. We will aim to accomplish this in the full/final version of the paper.

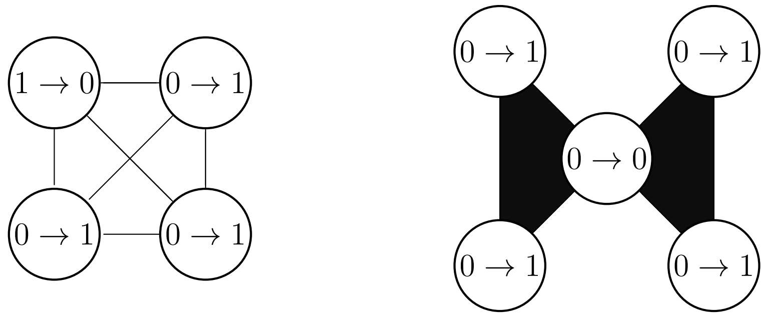

Figure 2: Unreachability in (a). a graph (b). a hypergraph with no graph edges. In each vertex the label of configuration w_1 is written first, that of w_2 next.

4 Upperbounding annihilation on hypergraphs

A particular setting where the previous result is applicable is given by our motivating examples: the XOR-SAT problem if the system has a solution and the CTD for social balance. Therefore with these cases in mind we can define the hypergraph analogue of annihilation time:

Definition 9. Annihilation: *$c_{ann}(G)$ is the minimum $t \geq 0$ such that $A_i = 0$ for all i.*

Definitions 4 and 5 enable upperbounding annihilation on hypergraphs. It will be, however, easier to work in *continuous*, rather than discrete time. Instead of choosing (at each integer step) one random live node and an edge containing it, we will assume that the pairs consisting of live nodes and associated edges get activated according to a Poisson process of an appropriate rate (see [AF14]). We have:

Theorem 2. *Suppose G is a hypergraph without graph partitions and w_1 is a configuration such that the a.r.w. on G can reach annihilation. Then one can couple the coalescing and annihilating random walks on G such that $c_{ann}(G) \leq c_{coal}(G)$.*

Proof. We will define the following stochastic process P:

1. **Initial state:** $A_i = \{(i, \infty)\}$. Note that $A_1 \cup A_2 \cup \ldots \cup A_n = [n] \times \infty$ and that each A_i contains at most one index b_i with $(b_i, \infty) \in A_i$.

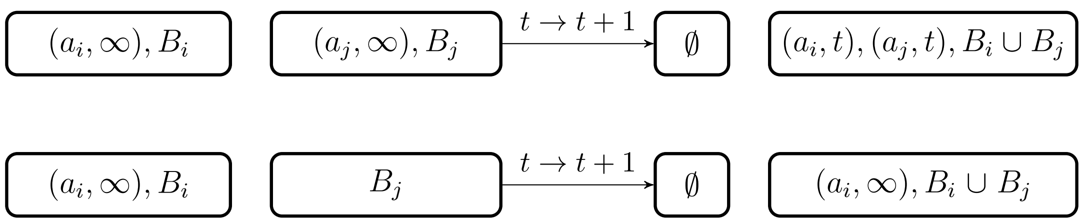

Figure 3: The two cases of stochastic process P. Only two nodes inside a common hyperedge are pictured.

We will call such a set *live* and b_i *the witness* for A_i. Also denote $B_i = A_i \setminus \{(i, \infty)\}$ if i is live, $B_i = A_i$ otherwise.

2. **Move:** At time t: Choose random vertex i (not necessarily live). Choose random edge $(i, j_1, \ldots, j_k)$. For $r = 1, \ldots, k$

   - If both A_i, A_{j_r} are live then make $A_{j_r} = (B_i \cup B_{j_r}) \cup \{(b_i, t), (b_{j_r}, t)\}$.
   - If, on the other hand, at most one of A_i, A_{j_r} is live then make $A_j := A_i \cup A_{j_r}$.

   Finally make $A_i = \emptyset$. Note that if we "move" a dead set A_i to a live set A_j then A_j will still be live.

3. **Stopping:** *Stopping time* $c_P(G)$ is the minimum $t \geq 0$ such that at most one i is live (one if n is odd, none if n is even)

Claim 1. *The following are true:*

1. P **observed on** $[n] \times \infty$ **and moves of live sets only** *is the annihilating random walk on* G. *If* n *is even then at time* $c_P(G)$ *all particles have annihilated. Consequently* $c_{ann}(G) \leq c_P(G)$.

2. *P where we disregard second components in all pairs is identical to the coalescent random walk on G and $c_P(G) = c_{coal}(G)$.*

A "proof by picture" is given in Figure 3. There are two cases: j is live or not. In both cases the observed process is identical to the annihilating random walk. Note that if n is even then when coalescence occurs in the c.r.w. all particles have died in the a.r.w.

□

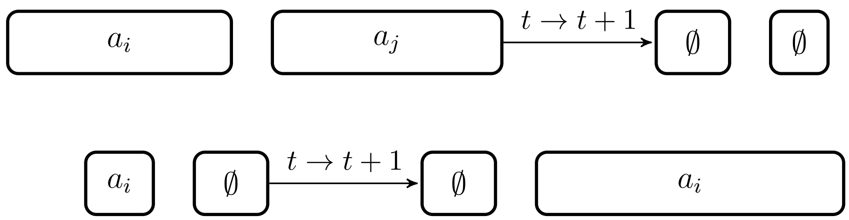

Figure 4: First coupled version: annihilating random graphs (the two cases). Only two nodes inside a common hyperedge are pictured.

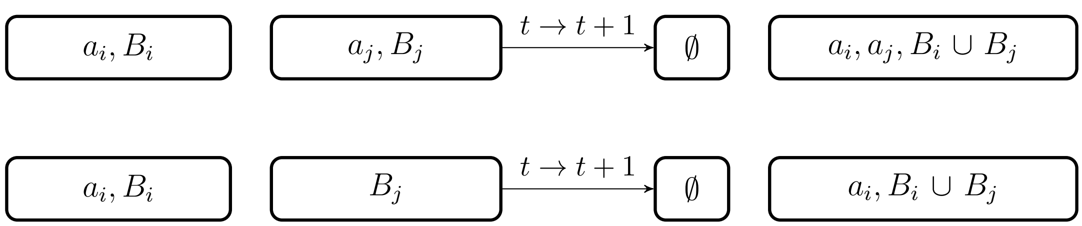

Figure 5: Second coupled version: coalescing random walks (the two cases). Only two nodes inside a common hyperedge are pictured.

The a result such as the previous one is interesting is that on graphs (see [AF14]) $c_{coal}(G)$ is identical (via duality) to coalescence time of voter model $c_{VM}(G)$, which can in turn be upper bounded in terms of a so-called *Cheeger coefficient of graph G*, essentially the inverse of the more well-known Cheeger constant of G. Similar results holds on hypergraphs:

Theorem 3. *For any hypergraph H the coalescence time $c_{coal}(H)$ and the parity time of the associated voter model $c_{VM}(H)$ are identically distributed.*

Proof. The proof is an adaptation of the classical duality argument [AF14]: we will define a process on *oriented hyperedges* in H (that is edges with a distinguished vertex) that will be interpreted in two different ways: as parity in the voter model and coalescence in the coalescent random walk.

The process is described in Figure 6. There is a certain difficulty in drawing pointed events in hypergraphs. In the figure we chose (in the interest of readability) not to represent the hyperedges vertically, but as triangles with a spatial extent, instead marking on the time axis the moment the given hyperedge event occurs (times t_1 and t_2 in the coalescing random walk). Horizontal lines (e.g. for ball 3 between moments t_1 and t_2) refer to histories not interrupted by any hyperedge event between the corresponding times.A horizontal line may be interrupted by a hyperedge event pointed at the given node.

A *left-right path P* between node i and node j is a sequence of hyperedge events and horizontal lines such that:

- P starts with a horizontal line of node i and ends with a horizontal line of node j.
- Every horizontal line of a node is followed by a hyperedge event with the corresponding node being pointed.
- Every hyperedge event is followed by an unique horizontal line corresponding to a **nonpointed node**.

For instance, in the picture from Figure 6 we have represented three left-right paths, between node 2 and each of nodes 1,4,5.

In the c.r.w. the activation of a hyperedge $e = [j \rightarrow i_1, i_2, \ldots i_r]$ pointed at vertex j is interpreted as vertex j being chosen (together with edge e), thus sending a copy of its cluster of balls to all other neighbors.

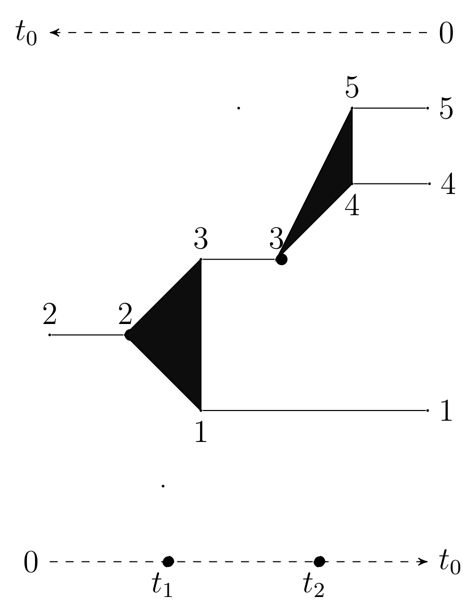

Figure 6: Coupling the coalescing random walk and the voter model. Time runs from left to right in the coalescing random walk and right to left in the voter model. At time t_1 (in the c.r.w.) copies of balls at (pointed) node 2 are sent to nodes 1 and 3. Similarly, at time t_2 copies of cluster at (pointed) node 3 are sent to nodes 4 and 5.

In the voter model the activation of a hyperedge $e = [j \to i_1, i_2, \ldots i_r]$ pointed at vertex j is interpreted as j adopting the multiset union of opinions of $i_1, i_2, \ldots, i_r$.

For instance, in the picture in Figure 6:

- in the c.r.w., assuming that initially $A_i = \{i\}$, $i = 1, 5$, at moment t_0 we have $A_1 = \{1, 2\}$, $A_2 = \emptyset$, $A_3 = \emptyset$, $A_4 = \{2, 3, 4\}$, $A_5 = \{2, 3, 5\}$.

- in the voter model at moment t_0 we have $A_1 = \{1\}$, $A_2 = \{1, 4, 5\}$, $A_3 = \{4, 5\}$, $A_4 = \{4\}$, $A_5 = \{5\}$. Label 3 has disappeared from the system.

Just as in the ordinary c.r.w./voter model, the existence of a left-right path between nodes i and j (e.g. $(2, 1), (2, 4), (2, 5)$) is interpreted as the event:

- In the c.r.w.: "at time t_0 node j holds a ball with label i."
- In the voter model: "at time t_0 node i holds opinion j with multiplicity at least one."

Moreover one path may contribute (when it does) with *exactly one ball/opinion* of a given type.

Consider now the event: "at t_0 every node on the right-hand side is connected to nodes on the left-hand side by an even number of paths".

- In the coalescing random walk this is equivalent to "at t_0 we have coalescence".
- In the voter model this is equivalent to "at t_0 we have parity of opinions"

□

Parzachevski et al.[PRT12] have given an extension of the Cheeger constant to simplicial complexes. It turns out that we need a related but slightly less demanding "odd Cheeger constant" notion:

Definition 10. *For a k-regular hypergraph H define* coefficient τ_H *as* $\tau_H = \sup_{0<|A|<|V|} \frac{k|A||\overline{A}|}{n \cdot |E(A,\overline{A})|}$, *where $E(A,\overline{A})$ is the set of all edges e of size at least two, with an odd number of vertices in A and all other vertices in $\overline{A}$.*

The next result is useful only in hypergraphs for which $E(A,\overline{A}) \neq \emptyset$ for all $0 < |A| < |V|$. In that case:

Theorem 4. *$E[c_{VM}(H)] \leq 2n\tau_H \cdot \ln(2)$.*

Proof. Consider a partition of the vertices of V into two parts, B and $\overline{B}$ and consider the following process, similar to the "two party voter model" from [AF14]:

- At time $t = 0$ start the process with 0 on labels of vertices of B ("reds") and 1 on vertices of $\overline{B}$ ("blues").
- choose a random vertex v and a random edge $e \ni v$ and we let $A_v = \sum_{w \neq v \in e} |A_w| \pmod 2$.
- We denote by N_t^B the number of vertices that have label 0 at time t.

- Denote by C^B the first time when $N_t^B \in \{0, N\}$, where N is the number of vertices of G.

N_t^B decreases by one exactly when the vertex chosen v has label 0 and the edge $e \ni v$ contains an odd number of nodes with label 1. On the other hand it increases by one precisely when the vertex chosen v has label 1 and the edge $e \ni v$ has an odd number of nodes with label 1. Thus

$$Prob[N_{t+dt}^B - N_t^B = 1] \geq \frac{1}{\tau_H} \cdot \frac{N_t^B \cdot (n - N_t^B)}{n}, \text{ and}$$
$$Prob[N_{t+dt}^B - N_t^B = -1] \geq \frac{1}{\tau_H} \cdot \frac{N_t^B \cdot (n - N_t^B)}{n}.$$

By an analysis similar to the case of voter models on graphs [AF14]

$$E[C^B] \leq n\tau_H \cdot \ln(2).$$

Now note that state where all vertices have an odd number of balls cannot be reached (is a so-called *garden of Eden*) in the coalescing random walk. This corresponds by duality to the state where every opinion is present an odd number of times in the system (in the voter model) also being a garden of Eden, hence unreachable if $B \neq \emptyset$, $B \neq V(G)$. So the event measured by $E[C^B]$ is really parity of opinions.

We complete the rest of the proof along the lines of the corresponding argument for graphs in [AF14]. □

5 Applications

Putting the last three inequalities together, applying them to k-XOR-SAT and getting back from a continuous to an equivalent discrete time model we get the following upper bound on convergence time of $RandomWalk$ on solvable instances H of k-XOR-SAT whose dual $D(H)$ is a simple hypergraph:

Theorem 5. *Let Φ be a satisfiable instance of XOR-SAT such that $H = D(\Phi)$ is simple. Then*

$$E[RandomWalk] \leq 2m^2\tau_{D(H)} \cdot \ln(2)$$

where m is the number of equations in H.

Details and many more results (e.g. upper bounds on annihilation similar to those in [CEOR13]) should be a subject for the journal-length version of this paper.

6 Conclusions, open problems and Acknowledgments

It would be interesting to see if the running time of other local search procedures, perhaps for more interesting problems like k-SAT can be analyzed in terms of (suitably defined) "particle systems".

This work has been supported by CNCS IDEI Grant PN-II-ID-PCE-2011-3-0981 "Structure and computational difficulty in combinatorial optimization: an interdisciplinary approach".

References

[AF14] D. Aldous and A. Fill. *Reversible Markov Chains and Random Walks on Graphs.* (book manuscript), Available from http://www.stat.berkeley.edu/~aldous/RWG/book.html, 2014.

[AKR06] T. Antal, P. L. Krapivsky, and S. Redner. Social balance on networks: The dynamics of friendship and enmity. *Physica D*, 224(130), 2006.

[ALL14] C. Avin, Y. Lando, and Z. Lotker. Radio cover time in hypergraphs. *Ad Hoc Netw.*, 12:278–290, 2014.

[CD12] J. Cooper and A. Dutle. Spectra of uniform hypergraphs. *Linear Algebra and its Applications*, 436(9):3268–3292, 2012.

[CEOR13] C. Cooper, R. Elsasser, H. Ono, and T. Radzik. Coalescing random walks and voting on connected graphs. *SIAM Journal on Discrete Mathematics*, 27(4):1748–1758, 2013.

[CFR13] Colin Cooper, Alan Frieze, and Tomasz Radzik. The cover times of random walks on random uniform hypergraphs. *Theoretical Computer Science*, 509:51–69, 2013.

[CT13] F. Chung and A. Tsiatas. Hypergraph coloring games and voter models. *Internet Math.*, 2013.

[DW83] P. Donnelly and D. Welsh. Finite particle systems and infection models. *Math.Proc. Cambridge Philos.Soc.*, 94:167–182, 1983.

[DW84] P. Donnelly and D. Welsh. The antivoter problem: Random 2-colorings of graphs. In B. Bollobás, editor, *Graph Theory and Combinatorics*, pages 133–144. Academic Press, 1984.

[Ist09] G. Istrate. On the dynamics of social balance on general networks (with an application to XOR-SAT). *Fundamenta Informaticae*, 91(2):341–356, 2009.

[Lig04] T. Liggett. *Interacting Particle Systems*. Springer, 2004.

[LP12] L. Lu and X. Peng. Loose laplacian spectra of random hypergraphs. *Random Structures & Algorithms*, 41(4):521–545, 2012.

[PRT12] O. Parzanchevski, R. Rosenthal, and R.J. Tessler. Isoperimetric inequalities in simplicial complexes. *Arxiv preprint arXiv:1207.0638*, 2012.

[RVYMO06] F. Radicchi, D. Vilone, S. Yoon, and H. Meyer-Ortmanns. Social balance as a satisfiability problem of computer science. *Physical Review E*, 026106, 2006.

[Sch14] J. Scherphuis. The mathematics of lights out. http://www.jaapsch.net/puzzles/lomath.htm, 2014.

[SKM12] J. Steenbergen, C. Klivans, and S. Mukherjee. A Cheeger-type inequality on simplicial complexes. *arXiv preprint arXiv:1209.5091*, 2012.