Gambler's ruin problem on Erdős-Rényi graphs

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Abstract

A multiagent ruin-game is studied on Erdős-Rényi type graphs. Initially the players have the same wealth. At each time step a monopolist game is played on all active links (links that connects nodes with nonzero wealth). In such a game each player puts a unit wealth in the pot and the pot is won with equal probability by one of the players. The game ends when there are no connected players such that both of them have non-zero wealth. In order to characterize the final state for dense graphs a compact formula is given for the expected number of the remaining players with non-zero wealth and the wealth distribution among these players. Theoretical predictions are given for the expected duration of the ruin game. The dynamics of the number of active players is also investigated. Validity of the theoretical predictions is investigated by Monte Carlo experiments.

Keywords: ruin game, random graphs, wealth distribution models

1. Introduction

Simple models of statistical physics and mathematics proved to be successful in understanding complex socio-economic phenomena [1, 2, 3]. Fluctuations of stock prices and stock indexes [4], wealth and income distribution [5], spreading of rumors [6] and infectious diseases [7], opinion formation [8] and voting [9], population statistics [10], human mobility [11], or the topology of various types of social networks [12, 13] are only a few well-known examples. Models developed to deal with these problems became interesting on their own for theoretical physicists and mathematicians. These models exhibit many times intriguing phase-transitions or lead to nontrivial spatiotemporal patterns and scaling laws. The techniques employed in their study rely on analytical and computational methods.

The "monopolist game" [15, 17, 16, 18] is one such model. It was first introduced by von der Malsburg [15] to explain the development of the orientation selectivity in the brain. Later Domingo and Watanabe, using the idea of the single winner [16], applied it to the study self-organization in neural networks. The model can be reformulated for studying wealth exchange and wealth distribution in a model society where individuals are connected through a social network. Here we consider such an attempt, and study a "monopolist game" type model on a simple random graph of Erdős-Rényi (ER) type [14].

The model studied in the present work is defined by the following dynamical rules: Let N_0 be the number of players participating in the game and connected through an ER type random network. Let L be the number of links in the network. At the beginning, all players have equal initial wealth, w_0. Players are allowed to play with each other only if they are directly connected by a link (first order neighbors on the graph) and if they have positive wealth. If on a link one player has no wealth it will become unable to participate in further games and its connections become inactive. In one time-step we allow one game on each active link, and the order in which the games are played is randomized. Players connected by the active link will put a unit from their wealth in a common pot, and then the pot is won randomly by one of them. If as a result of successive games one player loses all of its wealth it will be unable to play further games. The game is repeated until there are no two connected players who can play another game. We are interested in the time-length of the game as a function of the topological properties of the initial ER network (size and average number of links per node) and the statistics of the final wealth distribution. The rules for the model described above follows some ideas behind the monopolist game formulated in earlier studies.

Recently, Amano et al. [17] investigated ruin problems governed by rules similar in some aspects to the ones considered in the present study. They investigated a generalized ruin game with N_0 globally connected players, starting with some initial w_0 stake. In their game, in each round, from the N active players, there is one declared as winner who receives one unit. All the active players, put $1/N$ unit from their money to the pot, won by the winner with $1/N$ chance. Once a player remains with wealth less than $1/N$, he becomes inactive and must leave the game. His remaining wealth is evenly divided between the remaining players. The game ends when all the players became inactive but one, this will be the single winner, or monopolist. In order to calculate the length of the game, defined as the number of rounds played, they viewed the game as a multi-dimensional random walk with absorbing barriers. Their experimental and analytical results showed that the average length of the game has to be proportional to $(N \cdot w)^2$.

Following this path, E. Bach [18] considered a three player game, and derived an analytical solution, proving the conjecture made by Amano et al. Using martingale theory he showed an exact formula for the expected duration of the game. The idea to take the entrance fee to a game as one unit wealth is first mentioned in the paper by Amano et al. and was also investigated by E. Bach. They reported that the length of the game will be different, but this difference will be significant only for large number of players.

In the present study we consider thus a modified version of this ruin game, in order to make it more realistic for wealth exchange and wealth distribution in social systems. Instead of global connections, we consider an ER graph topology for the social interactions and set the entrance fee as one unit wealth for a game. Apart from the duration of the game, we will be interested also on the final statistics in wealth distribution.

The model we are concerned with is also related to the so-called *compulsive gambler model* of Aldous [19]. In contrast, however, to this latter model, a player only loses a limited amount as a result of a single game interaction.

2. Theoretical attempts

The ruin-game is performed on an ER graph, $G(N_0, p)$, defined by the number of nodes, N_0, and the p probability of having a link between two nodes. If we denote the number of links in the ER graphs as L, then $L = pN_0(N_0 - 1)/2$. The p probability defines also the average degree of a node, $\langle k \rangle = p \cdot (N_0 - 1) \approx p \cdot N_0$ (the later approximation is valid for large networks). The above formulas for L and $\langle k \rangle$ are valid in average only. The values of N_0 and p defines also the mean topological distance on the graph and it's diameter [14]. In order to have a one-component graph we need to have at least $p > \frac{(1+\epsilon)\ln N_0}{N_0}$ ($\epsilon > 0$) [20]. This condition ensures a connected graph in the thermodynamic limit. The results presented in our theoretical work are all statistical ones, averaged on the $G(N_0, p)$ ER graphs and different realizations of the dynamics.

2.1. Final-state approximation

The ruin game conserves the total wealth in the system ($w_t = N_0 w_0$). When the ruin-game is ended, only players that are not first order neighbors on the considered ER graph can survive. The dynamics is schematically illustrated in Figure 1.

The outcome of the game is a stochastic process; this allows to work with statistical averages. Our approximation is based on the hypothesis that the finally achieved states are equally probable, assuming that the conditions imposed by the rules of the game are satisfied. The first quantity we approximate under such assumptions is the average number of remaining nodes (N_R) in the final configuration. We argue than that the final wealth distribution should have an exponential form, and using this result we determine the expected maximum wealth, w_{max} for the nodes. The time-length of the game is determined by the rounds of ruin-games that are needed to gain/loose this maximum wealth amount. More specifically we assume a one dimensional Brownian motion approximation for the wealth of each node. Having 0 wealth as an absorbing state, we can estimate the total time of the game, by estimating the most probable time needed to reach this maximum wealth.

In the final configuration it is not possible to have two nodes with nonzero wealth connected by a direct link. Let us denote by N_s the average number of nodes in a $G(N_0, p)$ graph that fulfills the condition that there is no direct link between them. Due to the random nature of the wealth redistribution during the game we assume (**assumption 1**) that the average number of nodes with nonzero wealth in the final state (N_R) should be just this N_s value:

$$N_R \approx N_s \tag{1}$$

Definitely we have that for $N_s = 1$ (complete graph) we have $N_R = 1$ and for $N_s = N_0$ ($p = 0$) we have $N_R = N_0$. This assumption also means that each configuration of nodes that have no direct link between them is equally probable as

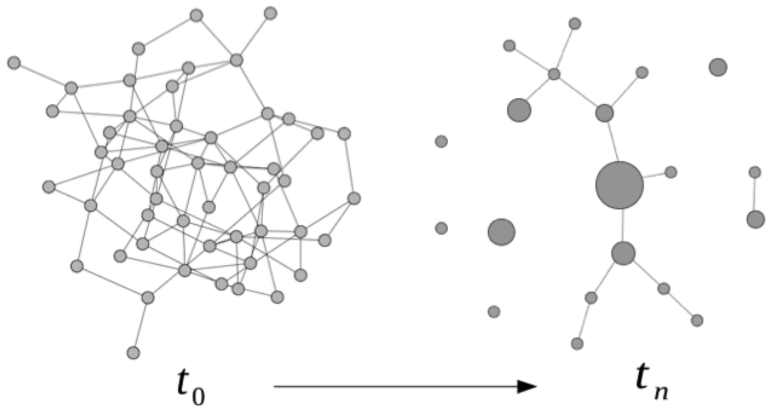

Figure 1: *The Erdős-Renyi random network (ER), generated with $< k >= 4$, at t_0 the beginning of the game, and after n time steps. Only nodes with wealth are shown on the figure, while the size of the nodes indicates their accumulated wealth.*

a final state. In a $G(N_0, p)$ ER graph the expected number of configurations where r nodes are not connected among each other by direct links is:

$$n(r) = C_{N_0}^{r}(1-p)^{r(r-1)/2} \tag{2}$$

The reasoning is based on the fact that for a $G(N_0, p)$ ER graph, the probability to have no connection between r selected nodes is $(1-p)^{r(r-1)/2}$. Using assumption 1, we get:

$$N_s = \frac{\sum_{r=1}^{N_0} n(r)r}{\sum_{r=1}^{N_0} n(r)} \tag{3}$$

The above sums cannot be computed analytically. In order to approximate them, instead of the N_s average value we can calculate the most probable value, N_p, and take this one as an approximation for N_s (**assumption 2**). Using the weights from equation (2), the most probable value can be determined by searching the maximum of $n(r)$. Using instead of $n(r)$, the value of $\ln[n(r)]$, the condition is:

$$\frac{d\ln[n(r)]}{dr}|_{N_p} = 0 \tag{4}$$

Using Stirling's formula and introducing the notation $x = N_p/N_0$, the condition (4) leads us to the equation:

$$\ln\left(\frac{1}{x}-1\right) - \frac{1}{2}\ln(1-p) + xN_0\ln(1-p) = 0 \tag{5}$$

Assuming now diluted graphs ($p << 1$) we assume $\ln(1-p) \approx -p$ end equation (5) writes as:

$$\ln\left(\frac{1}{x}-1\right) + \frac{p}{2} - x \cdot pN_0 = 0 \tag{6}$$

Neglecting the second term (independent of x) which for large graphs ($N_0 >> 1$) is much smaller than the last term, and introducing $\langle k \rangle = pN_0$ we get the equation:

$$\ln\left(\frac{1}{x}-1\right) - x \cdot \langle k \rangle = 0 \tag{7}$$

From this equation we learn that x should be independent both of w_0 and N_0. As result we get that x should depend only as a function of $\langle k \rangle$. Unfortunately equation (7) cannot be solved analytically, however we can give a fair analytical approximation for the solution using Taylor expansion. Keeping in mind that the binomial distribution alone would have its maximum at $x = 1/2$, we search the solution in this neighborhood and make a Taylor expansion up to first order for the logarithmic term around the $x = 1/2$ point:

$$\ln\left(\frac{1}{x} - 1\right) \approx -4\left(x - \frac{1}{2}\right) = 2 - 4x \tag{8}$$

In this approximation equation (7) leads us to:

$$x = \frac{1}{2 + \frac{\langle k \rangle}{2}} \tag{9}$$

This results shows that the Taylor expansion in (8) around the $x = 1/2$ point is justified only for small $\langle k \rangle$ values, ideally $\langle k \rangle < 1$. In this limit, however, the graph is not a one-component graph. For the realistic situation $1 < \langle k \rangle << N_0$, a better approximation is achieved if instead of $x = 1/2$ we perform in (8) a Taylor expansion around the $2/\langle k \rangle$ point (**assumption 3**):

$$\ln\left(\frac{1}{x} - 1\right) \approx \ln\left(\frac{\langle k \rangle}{2} - 1\right) - \frac{\langle k \rangle^2}{4 \cdot \left(\frac{\langle k \rangle}{2} - 1\right)}\left(x - \frac{2}{\langle k \rangle}\right) \tag{10}$$

The approximated solution of equation (7) writes as:

$$x = \frac{1 + \frac{\langle k \rangle - 2}{\langle k \rangle} \ln\left(\frac{\langle k \rangle}{2} - 1\right)}{\frac{3}{2}\langle k \rangle - 2} \tag{11}$$

For the most probable number of links we get thus:

$$N_p \approx N_0 \frac{1 + \frac{\langle k \rangle - 2}{\langle k \rangle} \ln\left(\frac{\langle k \rangle}{2} - 1\right)}{\frac{3}{2}\langle k \rangle - 2} \tag{12}$$

The analytic form (12) will be used from now on for $N_R \approx N_s \approx N_p$. According to this prediction in the final state the number of nodes with nonzero wealth does not depend on the initial wealth amount (w_0), and it scales linearly with the number of nodes in the graph (N_0).

The average wealth, $\langle w \rangle$ of the N_R remaining nodes is then:

$$\langle w \rangle = \frac{N_0 w_0}{N_R} \approx \frac{N_0 w_0}{N_p} = w_0 \frac{\frac{3}{2}\langle k \rangle - 2}{1 + \frac{\langle k \rangle - 2}{\langle k \rangle} \ln\left(\frac{\langle k \rangle}{2} - 1\right)} \tag{13}$$

Each node that remains with a finite final wealth has at least a unit amount of wealth. We assume now that all distributions of the $w_0 \cdot N_0 - N_R$ remaining wealth on the final N_R nodes are equally probable (**assumption 4**). This hypothesis is akin of our earlier hypothesis: assumption 1. It is easy to show that in the limit of large (when $N_0 >> 1$) graphs assumption 3 leads us to an exponential distribution for the final nonzero wealths (see Appendix 1)

$$\rho(w) \approx \alpha e^{-\alpha w}, \tag{14}$$

with $\alpha = 1/\langle w \rangle$.

The w_{max} maximal wealth can be approximated by the fact that in the final state there should be only one particle with wealth equal or bigger than w_{max}. Mathematically this writes as

$$N_R \cdot \int_{w_{max}}^{\infty} \alpha e^{-\alpha x} dx = 1, \tag{15}$$

leading to:

$$w_{max} = w_0 \frac{N_0}{N_R} \ln(N_R) \tag{16}$$

In order to obtain the expression for the expected duration of the monopolist game we need to determine the most probable time for reaching this threshold, assuming a simple Brownian motion approximation for the wealth with steps of one unit wealth.

The Distribution of the First Passage Time (DFPT) for a 1D Brownian motion [21] gives the probability to reach for the first time a limit x at time t, if at time $t = 0$ the motion started from coordinate x_0:

$$f(x_0, x_t, t) = \frac{|x_t - x_0|}{\sqrt{4\pi D t^3}} \exp(-\frac{(x_t - x_0)^2}{4Dt}) \tag{17}$$

D denotes the diffusion constant in one dimension

$$D = \frac{\epsilon^2}{2\delta t}, \tag{18}$$

with ϵ and δt the size and time-length of the steps, respectively.

The average value of the first passage time for an unbounded random walk cannot be calculated because the first moment of the DFPT diverges. In our case however the random walk is naturally bounded between 0 and $w_0 N_0$, so the average makes sense, and in computer experiments we do calculate these averages. In order to use however an analytical formula for the first passage time we consider the most probable value of it:

$$\tau_p = \frac{\Delta x^2}{6D} = \frac{|x_t - x_0|^2}{6D} \tag{19}$$

We assume now that the average duration of the game can be approximated with it's most probable value (**assumption 5**). If this holds the duration of the game is determined by the length of the game between the last two surviving and connected pair. These players are assumed to gather during previous games the maximum amount of wealth, and the total pot of w_{max} will be the final wealth of the winner. In such view now we assume (**assumption 6**) that the length of the game should be proportional with the first passage time from w_{max} to 0 in a 1D Brownian motion with $\delta t = 1$ and $\epsilon = 1$. This estimates the duration of the game as:

$$\tau \approx \frac{1}{3} w_{max}^2 \tag{20}$$

Taking into account our estimate for w_{max} we get

$$\tau \approx\approx \frac{w_0^2}{3} \left[\frac{N_0}{N_R} \ln(N_R) \right]^2 \approx \frac{w_0^2}{3} \left[\frac{N_0}{N_p} \ln(N_p) \right]^2, \tag{21}$$

leading to:

$$\tau^{1/2} \approx \frac{w_0}{\sqrt{3}} \frac{\frac{3}{2}\langle k \rangle - 2}{1 + \frac{\langle k \rangle - 2}{\langle k \rangle} \ln\left(\frac{\langle k \rangle}{2} - 1\right)} \left[\ln(N_0) + \ln\left(\frac{1 + \frac{\langle k \rangle - 2}{\langle k \rangle} \ln\left(\frac{\langle k \rangle}{2} - 1\right)}{\frac{3}{2}\langle k \rangle - 2} \right) \right] \tag{22}$$

This results suggests that $\tau^{1/2}$ scales linearly with the initial wealth (w_0) and it has a linear dependence as a function of the logarithm of the graphs size ($\ln N_0$). The dependence as a function of the average connectivity of the graph is more complex.

2.2. Time-evolution approximation

There is also a possibility to consider evolution equations for the number of active nodes and connections. Due to the involved approximations which are necessary for an analytic solution, this approach is suitable only for the first part of the dynamics and therefore the statistics of the final state and the time-estimate for the game cannot be obtained.

We treat the monopoly game as a process of elimination of the active connections between the nodes. This process continues until there is none left. In such an approach we trace the conditions under which connections are destroyed and the evolution equations will give the variation of some statistical parameters of the system as a function of time. We introduce the following variables: $N(t)$ - the number of active nodes (nodes which have wealth and are connected to at least another node with nonzero wealth, $N(0) = N_0$) at time t, $N_i(t)$ - the number of nodes that have wealth but are isolated, $N_n(t)$ - the number of nodes left without wealth, $N_c(t)$ - number of active connections. (naturally: $N(t) + N_i(t) + N_n(t) = N_0$). Our approximation is based on the assumption that $N_i << N_n + N$. Under such conditions:

$$\frac{dN_n}{dt} \approx -\frac{dN}{dt} \tag{23}$$

An active connection is destroyed when one of its nodes looses its wealth. When a node with wealth looses its wealth, it eliminates all its active connections. In a mean-field like approximation we can thus write:

$$\frac{dN_c}{dt} = -c(t)\frac{dN_n}{dt} = c(t)\frac{dN}{dt} \tag{24}$$

where $c(t) = \frac{2N_c(t)}{N(t)}$ is the average number of active connections per node at time t.

Each node performs a 1D random walk in its wealth, as described in the previous chapter. We can consider these random walks to be independent, since with large numbers of nodes and connections each node will likely be trading with different neighbors. If at time t the average wealth of the active nodes is $w(t)$, according to equation (19) it's most probable life-time is given by the most probable first passage time

$$\tau = w^2(t)/3, \tag{25}$$

where we have used $\epsilon = \delta t = 1$, and consequently $D = 1/2$. The number of nodes without wealth will increase thus by unity after τ number of steps. The time-unit was however fixed in our model as, $N_c(0) = L$ trials (one game on each links of the $G(N, p)$ ER, graph). The probability to hit an active connection, while selecting the $N_c(0)$ links is: $N_c(t)/N_c(0)$. According to these, we can approximate the rate at which the number of nodes without wealth increases:

$$\frac{dN_n}{dt} = \frac{3}{w^2(t)}\frac{N_c(t)}{N_c(0)}N_c(0) = N_c(t)\frac{3}{w^2(t)} \tag{26}$$

Since we neglected the isolated nodes in our approximation, the average wealth of the active nodes is easily computable:

$$w(t) \approx \frac{N_0 w_0}{N(t)} \tag{27}$$

We have thus two coupled differential equations for describing the dynamics of the system:

$$\begin{aligned} \frac{dN_c}{dt} &= \frac{2N_c(t)}{N(t)}\frac{dN}{dt} \\ \frac{dN}{dt} &= -\frac{3N(t)^2}{N_0^2 w_0^2}N_c(t) \end{aligned} \tag{28}$$

The first differential equation in this system is separable, leading to:

$$N_c(t) = \frac{L}{N_0^2}N(t)^2 \tag{29}$$

Plugging this solution into the second equation in (28), we get again a separable first order differential equation, with the solution:

$$N(t) = N_0\left[\frac{9tL}{N_0 w_0^2} + 1\right]^{-1/3} \tag{30}$$

which leads to:

$$N_c(t) = L\left[\frac{9tL}{N_0 w_0^2} + 1\right]^{-2/3} = N_0\frac{\langle k\rangle}{2}\left[\frac{9t\langle k\rangle}{2w_0^2} + 1\right]^{-2/3} \tag{31}$$

We keep in mind again, that these predictions are valid only for the first part of the dynamics, where the number of isolated nodes is small. The predicted dynamics for $N(t)$ and $N_c(t)$ cannot be used thus for estimating the final state of the system or the time-length of the game. Comparison of theoretical predictions with simulated data in Fig. 7 confirms these hypothesis.

3. Computer experiments

The ruin game can be easily studied by Monte Carlo (MC) type simulations. Simulations have both the role to confirm the theoretical predictions and/or to verify the assumptions used in our calculations (assumptions 1-6).

First, an Erdös-Rényi type $G(N_0, k)$ ($k = p \cdot N_0$) graph was generated, with all the nodes having equal initial assets w_0. Only one-component graphs were used in the MC simulations. The game began with selecting randomly one connection from the list of $N_c(0) = L$ initial connections (edges). The players in this game were the two nodes connected by the selected link. Both of the nodes lost one unit from their wealth, then one of them was selected by chance (with 1/2 probability) to win the pot. The wealth of the winning node was increased by two units and time was increased by $1/N_c(0)$. In case, one node remained without assets, it was declared loser and was deleted from the network with all of its connections. By repeating this sequence, the size of the network and the number of total connections began to decrease, and after several rounds isolated nodes started to appear. In each further simulation step we allow a game on each active link. The active links are ordered in a random sequence, and the games are performed respecting this order and immediately deleting all links that become inactive after a game.

In order to calculate averages and to get reasonable statistical results, for each set of input parameters, simulations were repeated on 10 randomly generated graphs and for each of these graph we considered 100 independent dynamics for the ruin game. The averages presented here are calculated for the the whole ensemble of $10 \times 100 = 1000$ realizations if it is not specified in another way. We tracked the total number of time-steps needed to reach the final configuration, the number of nodes remaining with a finite wealth and their wealth distribution. In order to get experimental results for the theoretically predicted trends we varied the system size (N_0), initial wealth (w_0) and average degree of the nodes ($\langle k \rangle$) in the ER network. By increasing the value of N_0, $\langle k \rangle$ and w_0, in agreement with our theoretical prediction (eq. (22)) the length of the simulations were also increasing and therefore we were able to get good statistics only for $N_0 \leq 500$, $w_0 <= 10$ and $\langle k \rangle \leq 50$.

3.1. Checking the theoretical assumptions

In our "final state approximation" we have used 6 assumptions. These assumptions formulate also important results for the ruin game. Checking the validity of our assumptions provides also a possibility to improve our theoretical methods. We can detect in such manner the steps that introduce the largest errors in the theory and, consequently, one can work on these points to improve further the theoretical description. We now verify the assumptions by using MC simulations.

In **assumption 1** we stated that the number of remaining links is a random variable, that can be approximated as the size of randomly selected sets of nodes that are not directly connected among them in the $G(N_0, p)$ graph. This assumption leads us to $N_R \approx N_s$, with N_s given in equation (3). Typical simulations results for N_R as a function of the initial wealth (w_0), system size (N_0), and the average number of links per node ($\langle k \rangle$) are shown with open circles in Figures 2a-d. According to equation (3) the remaining number of nodes should not depend on the value of the initial wealth w_0 . Fig 2a shows that the simulated N_R values are indeed independent of the w_0 values. From Fig. 2b we learn that N_s is a good approximation for the N_R value. Figure 3 shows that the distribution of the N_R values is roughly Gaussian and is in excellent agreement with the distribution of the $n(r)$ weight in equation (2), confirming again our assumption 1.

Assumption 2 stated that the N_s average value can be approximated by the N_p most probable value. Figure 3 proves this, since we show that both N_R and N_s has a normal distribution.

Our hypothesis that N_p is a good approximation for N_r is based also on **Assumption 3**, according to which the analytical formula (11) is a good approximation for the solution of equation (7). Both equations [(11) and (7)] suggests that N_p should be linearly dependent as a function of N_0 and should be independent of w_0. In order to show that equation (11) indeed gives a good solution, we have to show that the $N_p(\langle k \rangle)$ dependence obtained from (11) and (7) are similar. In Fig. 2c we plotted with dotted line the numerical solution for N_p obtained from (7) and with a continuous line the approximated solution (12) as a function of $\langle k \rangle$. The two curves matches excellently, and gives a good approximation for N_s also. Our final approximation for N_p given by equation (12) gives also a good approximation for the simulated N_R values. This can be seen on the Figures 2a-c.

Assumption 4 states that all distributions of the $w_0 \cdot N_0 - N_R$ remaining wealth on the final N_R nodes are equally probable, leading (see the Appendix) to an exponential distribution of the wealths of the nodes that remain with nonzero wealth. This assumption can be proved again by MC simulations. On Figure 4 we present typical results in such sense.

According to **assumption 5**, the average duration of the game can be approximated with it's most probable value. On Fig. (5)a we prove this hypothesis by plotting the simulated mean and most probable duration of the game as a function of the graph size N_0. The two values matches well, and one can see that the mean value has much smaller fluctuations, so it makes sense to compute this value in simulations.

Finally **assumption 6** states that the duration of the game can be estimated as the first passage time of a 1D Brownian motion from w_{max} to 0. From this hypothesis it results a connection between the simulated τ and N_R values, equation (21), which can be easily verified. On Figure 6 we plot the simulation results (circles) together with the prediction of equation (21). The excellent agreement shows that assumption 6 is justified.

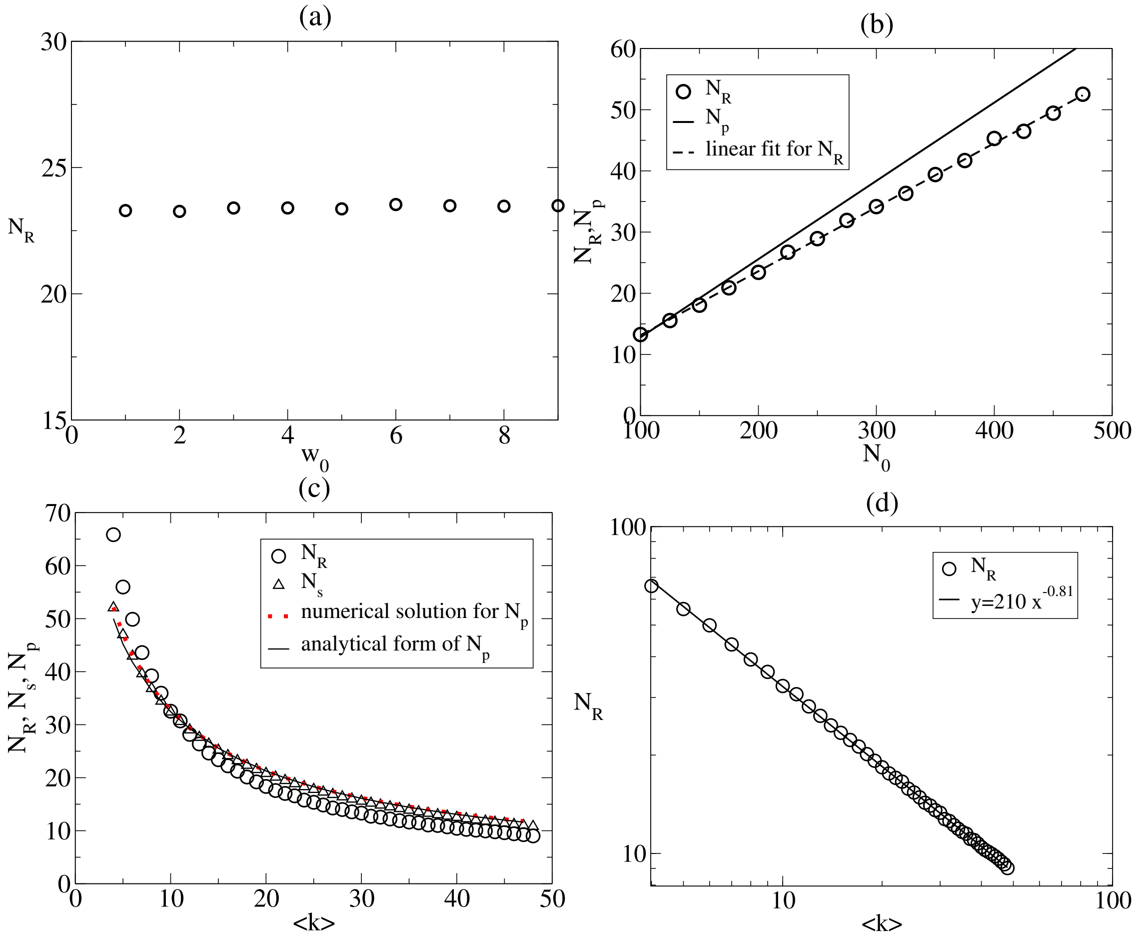

Figure 2: (color online) *Simulation results for the average number of remaining nodes.* (*a*) *Results as a function of the initial wealth of the nodes,* w_0 *(*$\langle k \rangle = 15$ *and* $N_0 = 200$*);* (*b*) *Results as a function of the initial size of the graph,* N_0*. The continuous line indicates the* N_p *value given by equation (12) and with dashed line we indicate the best linear fit for the simulation results. (*$\langle k \rangle = 15$, $w_0 = 4$*);* (*c*) *Results as function of the average number of links per node,* $\langle k \rangle$*. The circles indicate the simulation results, while triangles show the value of* N_s *computed from equation (3). The dotted line indicates the numerical solution for* N_p *obtained from equation (7) and the continuous line corresponds to the approximation given by equation (12). (*$N_0 = 200$, $w_0 = 4$*);* (*d*) *Power-law fit for the simulation results presented in figure* (*c*) *. Please note the logarithmic axes.*

3.2. Checking the theoretical predictions

Our "final state approximation" focused mainly on predicting the number of nodes that remain with nonzero wealth (N_R) and the τ duration of the ruin game. The main result for $N_R \approx N_p$ is given by equation (12). The predictions of this equation are verified by MC simulations and the results are summarized in Figures 2a-c. According to these

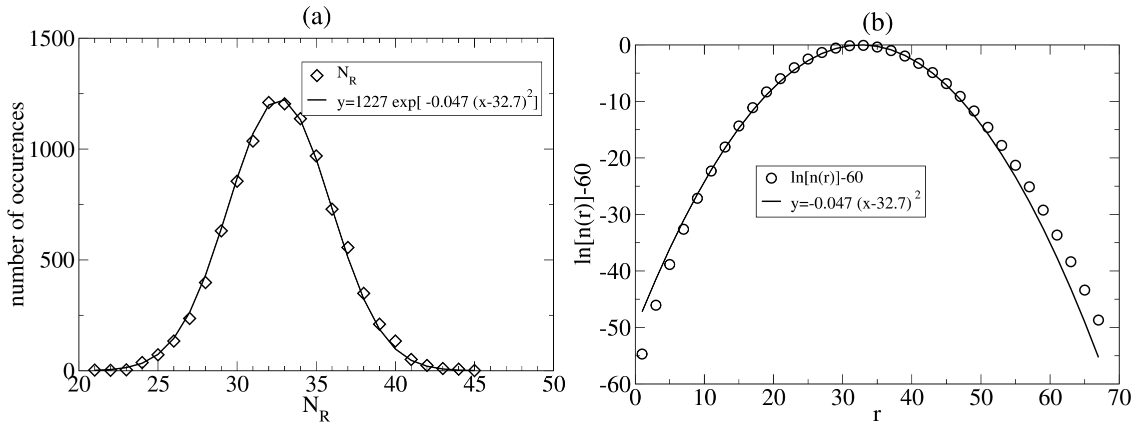

Figure 3: *(a) Simulation results for the distribution of the remaining number of nodes. The solid line indicates a Gaussian fit with* $y = C\ exp[-0.047(x-32.7)^2]$. *(*$N_0 = 200$, $\langle k \rangle = 10$, $w_0 = 4$ *and 10 000 trials) (b). The value of* $n(r)$ *calculated from equation (2) and fitted with the very same Gaussian as the simulations results.*

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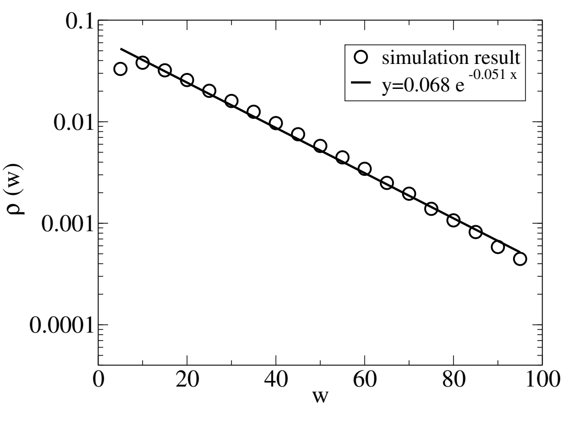

Figure 4: *Exponential distribution of wealth for the nodes that remain with finite wealth at the end of the game. Please note the logarithmic axis for* $\rho(w)$. *Circles are simulation results, while the continuous line is an exponential fit:* $\rho(w) = 0.068 \cdot \exp(-0.051w)$. *(*$N_0 = 200$, $\langle k \rangle = 10$ *and* $w_0 = 4$*).*

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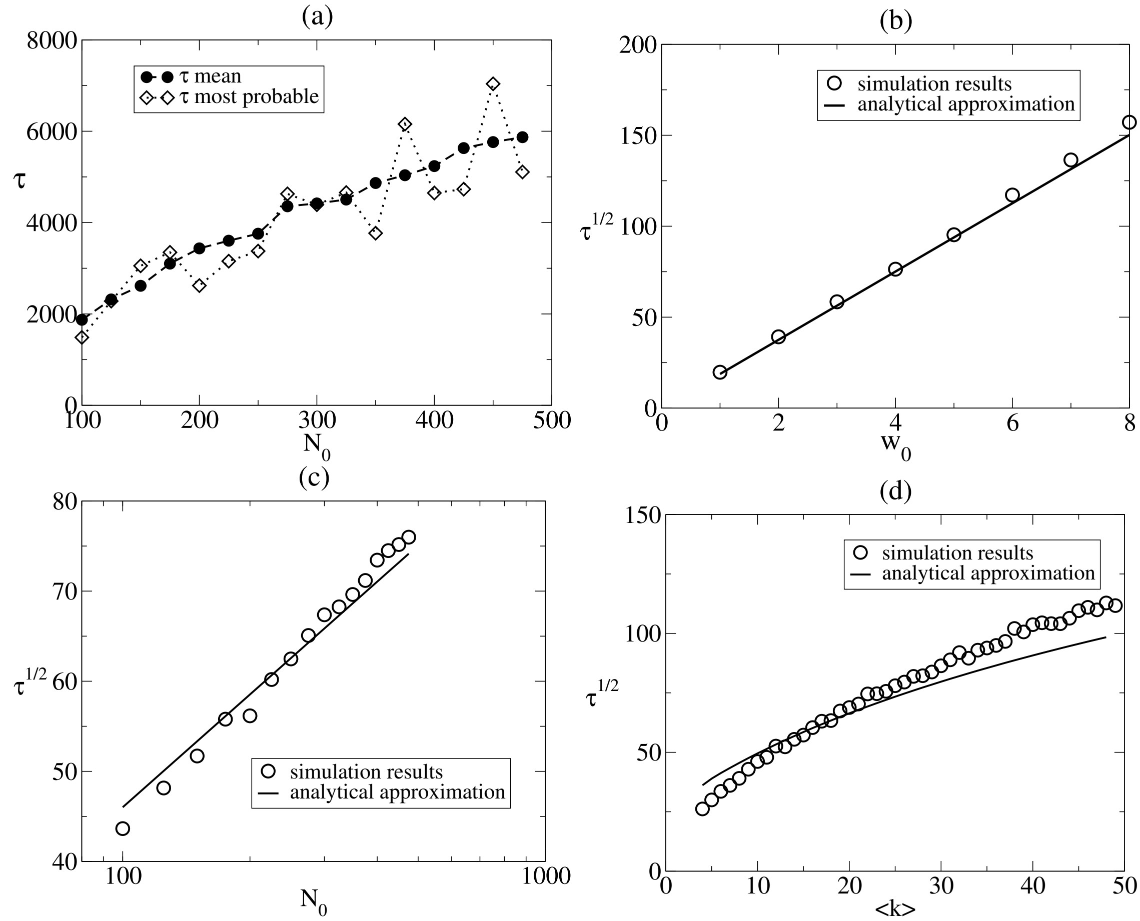

Figure 5: *Simulation results for the time-length of the ruin game.* (a) *Comparison between the mean and most probable length of the game.* ($\langle k \rangle = 15$ *and* $w_0 = 4$)*;* (b) *Square root for the games time-length as a function of the nodes initial wealth* w_0 *(circles)* ($\langle k \rangle = 15$ *and* $N_0 = 500$)*;* (c) *Square root of game length as a function of the initial size of the graph,* N_0 *(circles).* ($\langle k \rangle = 15$ *and* $w_0 = 4$)*. Please notice the logarithmic axis for* N_0*;* (d) *Square root of the games length as a function of the mean degree of the nodes,* $\langle k \rangle$*.* ($w_0 = 4$ *and* $N_0 = 200$)*. For figures (b), (c) and (d) the continuous line indicates our analytical approximation (22).*

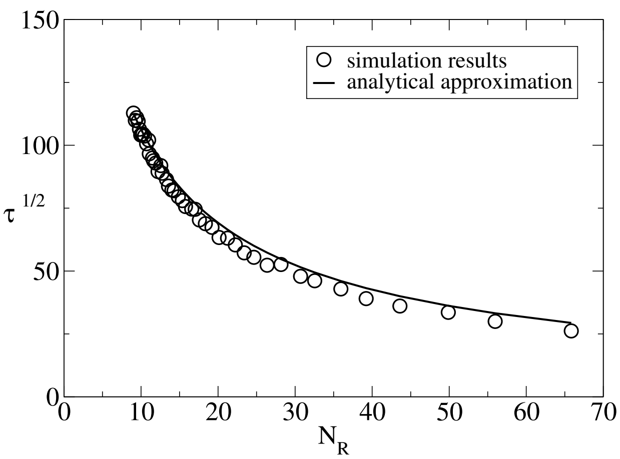

Figure 6: *Square root for the game length as a function of the number of nodes with nonzero wealth in the final configuration. Circles are simulation results both for* τ *and* N_R*, while the continuous line is given by equation (21).* ($N_0 = 200$*,* $w_0 = 4$ *and* $\langle k \rangle \in [4, 50]$)*.*

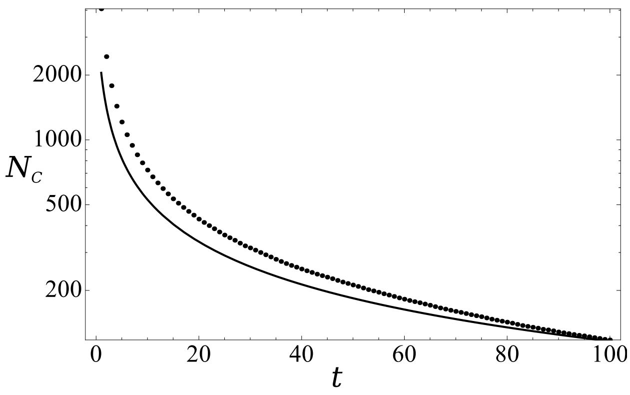

Figure 7: *Time evolution of the number of active connections in the system (N_c) as the game progresses. The solid line is the theoretical prediction given by equation (7) and the simulation results are marked with dots.* $N_0 = 1000$, $< k >= 10$, $w_0 = 4$, *results averaged across 100 trials.*

figures we can conclude that the simple approximation (12) works surprisingly well. The fact that N_R is independent of w_0 and scales linearly with N_0 is nicely confirmed. The dependence as a function of $\langle k \rangle$ is much more complicated but it can be approximated by equation (12). A better analytical approximation can be given for $N_r(\langle k \rangle)$, realizing that MC simulations suggest a power-law dependence (Fig. 2d) with an exponent of ≈ -0.81.

Our prediction for the duration of the ruin game on the $G(N_0, p)$ graph is given by equation (22). According to this we get a very simple linear trend for $\sqrt{\tau}$ as a function of w_0. This can be confirmed immediately by MC simulations (see Fig. 5b). Surprisingly, not only the linear trend is reproduced, but the actual values predicted by equation (22) are in good agreement with the simulation results. Studying now the trends of $\sqrt{\tau}$ as a function of N_0 and $\langle k \rangle$ leads us to the same conclusion regarding the validity of equation (22). Plotting the value of $\sqrt{\tau}$ as a function of the logarithm of the system size ($\ln N_0$), the MC simulations reproduce nicely the linear trend (Figure 5c). Again, the actual values given by equation (22) are also in good agreement with simulation results. The trend of $\sqrt{\tau}$ as a function of $\langle k \rangle$ is much more complicated [see equation (22)], but again the agreement between MC simulation data and our theoretical prediction is surprisingly good. Figure 5d illustrates this agreement for a fixed N_0 and w_0 value.

Now that the validity of our final formula (22) for the ruin game duration is confirmed by the MC experiments, we test the prediction given for the time evolution of the system. More precisely we check the time evolution equation for the number of active links in the system as a function of the simulation time [equation (31)]. For a fixed N_0, $\langle k \rangle$ and w_0 value the time evolution of $N_c(t)$ obtained by MC simulations is plotted against the prediction (31) on Figure 7. Although not perfect, the theoretical predictions for the first part of the dynamics seems to be fair. As expected however, in the last part of the dynamics where the number of isolated nodes is not negligible the agreement is less good. This picture suggests the limitations of our dynamical description, discussed already in the previous section.

4. Conclusions and Discussion

We have investigated theoretically and by MC experiments a ruin-game on ER graphs. Computer experiments confirmed the findings of our theoretical results. One of our most important result is that a compact approximation has been given for the N_R number of players that remain with non-zero wealth at the end of the game and for the τ time duration of the game.

We have shown that the number of remaining nodes is independent of the initial wealth of the nodes (w_0), it scales linearly with the system size (N_0), and the dependence asa function of $\langle k \rangle$ is complex according to equation (12), although simulations suggests a simple power-law trend.

For the time-duration of the ruin game on the $G(N_0, p)$ random graph our approximation given by equation (22) suggests that $\sqrt{\tau}$ scales linearly with the value of w_0 and it is a linear function of $\ln(N_0)$. According to equation (22) the dependence as function of the average degree of a node is more complex, and this formula gives a reasonable approximation.

We have also shown that the wealth distribution in the final state is an exponential distribution.

From our "final state approximation" we have learned that the final state of the system can be well approximated as a random selection of nodes in the $G(N_0, p)$ graph that are not directly linked, and the distribution of wealth on these nodes is also random: each distribution of the total wealth being equally probable.

A second approximation which deals directly with dynamical properties of the system was also used, and proved to be efficient for predicting the dynamics of the system's time evolution. However, the major shortcoming of this approach is its failure to work accurately at the end part of the game, where the number of isolated nodes becomes significantly large. This approximation fails to predict the final state of the system and the expected game duration.

The ruin-game considered in the present study can find several applications in modeling various socio-economic phenomena like: opinion formation, voting, clustering of a society according to some socio-economical preferences or even wealth (or customer) redistribution among economic agents.

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Appendix 1

By assuming that all different distributions for the $w_f = w_0 N_0 - N_R$ wealth units among the remaining N_R nodes is equally likely, one can easily estimate the probability that a remaining node has $n + 1$ wealth units (the minimal amount of 1 unit of wealth is assumed, since all remaining nodes has a nonzero amount of wealth, and in each game the wealth changes only by two units):

$$P(n+1) = \frac{\frac{[(w_f - n + (N_R - 2)]!}{(N_R-2)!(w_f-n)!}}{\frac{(w_f+(N_R-1)!}{(N_R-1)!w_f!}} \tag{32}$$

Let us consider now large ($N_0 >> 1$ and $N_R >> 1$) and dense graphs ($N_0 >> \langle k \rangle >> 1$) and study the $n << w_0 N_0$ limit of the $P(n)$ distribution. Using the Stirling formula for $\ln P(n+1)$, and approximating

$$\ln\left(1 - \frac{n}{w_0 N_0 - 2}\right) \approx -\frac{n}{w_0 N_0} \tag{33}$$

$$\ln\left(1 - \frac{n}{w_0 N_0 - N_R}\right) \approx -\frac{n}{w_0 N_0 - N_R}, \tag{34}$$

we get:

$$\ln P(n) \approx -\frac{n N_R}{w_0 N_0} + \frac{1}{N_R} O\left(\left(\frac{n N_R}{w_0 N_0}\right)^2\right) + F(w_0, N_0, N_R) \tag{35}$$

In the considered limits this leads to the assumed exponential wealth distribution.

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